NORTHEASTERN UNIVERSITY
Department of Mathematics

U165 Introduction to Math Reasoning **Final Examination** Fall 2008

1(10)	**2**(10)	**3**(10)	**4**(10)	**5**(10)	**Total**
6(10)	**7**(10)	**8**(10)	**9**(10)	**10**(10)	

Your name:
Instructor's name:
Andrei Zelevinsky

1. Is the statement $y \to \neg x$ logically equivalent to $\neg(x \wedge y)$?

Answer Yes - Here are the truth tables.

x	y	$x \to (\neg y)$
True	True	**False**
True	False	**True**
False	True	**True**
False	False	**True**

x	y	$\neg(x \wedge y)$
True	True	**False**
True	False	**True**
False	True	**True**
False	False	**True**

The truth table are identical - which shows the two logical expressions are equivalent.

2. How many integers in the range 1 to 400 (inclusive) are neither even, nor perfect squares? [*For example, in the range* 1 *to* 10*, there are only 3 such integers, namely,* 3, 5, 7.]

Answer 190

If U is the "universal" set $\{1, 2, ..., 400\}$, A is the subset of even integers in U, and B the subset of perfect squares, then the answer is

$$|U| - |A| - |B| + |A \cap B| = 400 - 200 - 20 + 10 = 190$$

3. Construct a bijection $f : \mathbf{N} \to \mathbf{N} \times \{0, 1, 2\}$, where $\mathbf{N}$ is the set of all nonnegative integers.

Answer

Observe, the image of each $n \in \mathbf{N}$ is an ordered-pair (x, y) with $x \in \mathbf{N}$ and $y \in \{0, 1, 2\}$.
Here is a solution.

$$f(n) = (n \ div\ 3, n \ mod\ 3)$$

As a table:

n	0	1	2	3	4	5	6	···
f(n)	(0,0)	(0,1)	(0,2)	(1,0)	(1,1)	(1,2)	(2,0)	···

4. How many social security numbers (that is, nine-digit strings)

(a) have at least one odd digit?

Answer $10^9 - 5^9$

Preliminary: The "complement" of "at least one" is "none". We can calculate the total number of nine-digit strings and subtract the number of strings which have no odd digits at all.

10^9 is the total number of strings of length nine where each symbol is chosen from the ten digits.
5^9 is the total number of strings of length nine where each symbol is chosen from the five odd digits.

(b) have exactly four 0's, and no repeated non-zero digits?

Answer $\binom{9}{4} \cdot (9)_5$

$\binom{9}{4}$: There are four "locations" out of nine locations to place the four 0's.

$(9)_5 = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5$: Choose from 9 non-zero digits for the first non-zero digit; then choose from 8 for the next; etc.

5. How long must be a binary string to guarantee that among its strings consisting of three consecutive terms there will be two identical ones?

Answer 11

Preliminary 0 - Example: Given a string of length 6 such as (a, b, c, d, e, f), we are examining the four substrings of length three $\{(a, b, c), (b, c, d), (c, d, e), (d, e, f)\}$.

Fact: If the original string is of length n, then there are $n - 2$ consecutive substrings of length three.

Preliminary 1: We are "counting" binary strings of length three; the symbols are chosen from $\{0, 1\}$.

Preliminary 2: There are a total of eight different binary strings $\{(0, 0, 0), (1, 0, 0), \cdots, (1, 1, 1)\}$.

If $n = 11$, then there are $n - 2 = 9$ consecutive substrings of length three. By the Pigeonhole principle, since there are only eight possible of strings of length three, it follows that two of the substrings are the identical.

6. Let the functions $f : \mathbf{Z} \to \mathbf{Z}$ and $g : \mathbf{Z} \to \mathbf{Z}$ be given by $f(x) = x^2 + 1$ and $g(x) = x^5$. Compute $(g \circ f)(x)$ and expand it as sum of powers of x with numerical coefficients, using Pascal's triangle.

Answer $x^{10} + 5x^8 + 10x^6 + 10x^4 + 5x^2 + 1$

$g \circ f(x) = g(x^2 + 1) = (x^2 + 1)^5$.

The Binomial Theorem says

$$(x^2 + 1)^5 = \sum_{k=0}^{5} \binom{5}{k} (x^2)^{5-k} (1)^k$$

Pascal's triangle allows us to obtain the coefficients $\binom{5}{k}$ easily. That is, the "fifth row" gives the coefficients.

The Fifth Row of Pascal's triangle is 1 5 10 10 5 1.

Putting these coefficients into the Binomial Theorem gives the answer.

7. Prove the following by induction: $1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \geq \frac{3}{2} - \frac{1}{n+1}$ for all $n \geq 1$.

Answer

The proof is by induction.

Basis case: Put $n = 1$. LHS $= 1$ and $RHS = \frac{3}{2} - \frac{1}{(1+1)} = 1$. $\Rightarrow 1 \geq 1$ ✓

Induction hypothesis: We want to prove the statement for $n = k + 1$. We assume the statement is true for $n = k$ with $k \geq 1$.

$$\text{Goal} \quad 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} \overset{?}{\geq} \frac{3}{2} - \frac{1}{(k+1)+1}$$

$$\begin{aligned}
\text{Start with} \quad & 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} & \\
\text{By Induction} \quad & 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} & \geq \frac{3}{2} - \frac{1}{k+1} + \frac{1}{(k+1)^2} \\
& & = \frac{3}{2} - \frac{k+1}{(k+1)^2} + \frac{1}{(k+1)^2} \\
& & = \frac{3}{2} - \frac{k}{k^2+2k+1} \\
\text{Using } k^2+2k+1 > k^2+2k \quad & & \geq \frac{3}{2} - \frac{k}{k^2+2k} \\
& & = \frac{3}{2} - \frac{1}{k+2} \quad ✓
\end{aligned}$$

8. Solve the recurrence relation $a_n = 3a_{n-1} + 5$ with the initial condition $a_0 = -2$.

Answer $\frac{1}{2} \cdot 3^n - \frac{5}{2}$

Solution is of the form $a_n = c_1 \cdot 3^n + c_2$ - where c_1, c_2 are to be found.

Calculating $a_1 = 3 \cdot (-2) + 5 = -1$.

$$\text{Solve} \begin{cases} -2 & = c_1 + c_2 \\ -1 & = c_1 \cdot 3 + c_2 \end{cases} \qquad \text{This gives } c_1 = \tfrac{1}{2} \text{ and } c_2 = -\tfrac{5}{2}.$$

9. Solve the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$, with the initial conditions $a_0 = 1$, $a_1 = 2$.

Answer $3^n - \frac{1}{3} \cdot n \cdot 3^n$

Associated quadratic is $x^2 - 6x + 9 = 0$.
This quadratic has a repeated root: $x = 3$

Solution is of the form $a_n = c_1 \cdot 3^n + c_2 \cdot n \cdot 3^n$.

Solve the system $\begin{cases} 1 &= c_1 \\ 2 &= c_1 \cdot 3 + c_2 \cdot 1 \cdot 3 \end{cases}$ This gives $c_1 = 1$ and $c_2 = -\frac{1}{3}$

10. Using Euclid's Algorithm, find integers x and y such that $29x + 24y = 1$.

Answer $x = 5$ and $y = -6$

Euclidean Algorithm:

a	b	a mod b	a div b
29	24	5	1
24	5	4	4
5	4	1	1
4	**1**	0	

Gives $GCD(29, 24) = 1$.

Because $GCD(29, 24) = 1$, it is possible to find x and y.

Solving Bottom-up

$$\begin{aligned} 1 &= 5 - 1 \cdot 4 \\ &= 5 - 1(24 - 4 \cdot 5) && \text{substituting} \\ &= -24 + 5 \cdot 5 \\ &= -24 + 5 \cdot (29 - 1 \cdot 24) && \text{substituting} \\ &= -6 \cdot 24 + 5 \cdot 29 \end{aligned}$$