NORTHEASTERN UNIVERSITY
Department of Mathematics

U165 Introduction to Math Reasoning **Final Examination** Fall 2008

1(10)	2(10)	3(10)	4(10)	5(10)	Total
6(10)	**7(10)**	**8(10)**	**9(10)**	**10(10)**	

Your name:
Solutions
Instructor's name:

1. Is the statement $y \to \neg x$ logically equivalent to $\neg(x \wedge y)$?

Truth table:

x	y	$\neg x$	$y \to \neg x$	$x \wedge y$	$\neg(x \wedge y)$
T	T	F	F	T	F
T	F	F	T	F	T
F	T	T	T	F	T
F	F	T	T	F	T

The truth values of the two statements are identical, so they are logically equivalent.

2. How many integers in the range 1 to 400 (inclusive) are neither even, nor perfect squares? [*For example, in the range* 1 *to* 10 *there are only* 3 *such integers, namely,* 3, 5, 7.]

Let $U = \{n \in \mathbb{Z} : 1 \le n \le 400\}$,
$A = \{n \in U : n \text{ is even}\}$
$B = \{n \in U : n \text{ is a perfect square}\}$.

Then $|U| = 400$, $|A| = 200$ (every second number is even), $|B| = 20$ (since $B = \{1^2, 2^2, \ldots, 20^2\}$), and $|A \cap B| = 10$ (since $A \cap B = \{2^2, 4^2, \ldots, 20^2\}$). So the answer is

$|U - (A \cup B)| = |U| - |A| - |B| + |A \cap B| = 400 - 200 - 20 + 10 = \boxed{190}$

3. Construct a bijection $f : \mathbf{N} \to \mathbf{N} \times \{0,1,2\}$, where $\mathbf{N}$ is the set of all nonnegative integers.

Recall: $\mathbb{N} \times \{0,1,2\}$ is the set of ordered pairs (q,r), where $q \in \mathbb{N} = \{0,1,2,3,\dots\}$, and $r \in \{0,1,2\}$.

Define f by the table

a	0	1	2	3	4	5	6	7	...
$f(a)$	(0,0)	(0,1)	(0,2)	(1,0)	(1,1)	(1,2)	(2,0)	(2,1)	...

Formula for f: $\boxed{f(a) = (a \text{ div } 3,\ a \bmod 3)}$,

or $\boxed{f(a) = \left(\left\lfloor \frac{a}{3} \right\rfloor,\ a - 3\left\lfloor \frac{a}{3} \right\rfloor\right)}$

4. How many social security numbers (that is, nine-digit strings)
(a) have at least one odd digit?

Total number of s.s.n. is 10^9

The number of those using only even digits is 5^9

Answer: $\boxed{10^9 - 5^9}$

(b) have exactly four 0's, and no repeated non-zero digits?

We have $\binom{9}{4} = \frac{9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 2}$ choices for the locations of four 0's.

After that, we have 9 digits (excluding 0) to fill the remaining 5 spots without repetitions: the number of choices is $(9)_5 = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5$.

Answer: $\boxed{\frac{9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 2} \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}$

5. How long must be a binary string to guarantee that among its strings consisting of three consecutive terms there will be two identical ones?

There are $8 = 2^3$ different binary strings of length 3. A binary string of length n has $n-2$ "substrings" consisting of three consecutive terms: they can be in places $1,2,3$, $2,3,4$, $3,4,5$, $\ldots$, $n-2, n-1, n$.

By the Pigeonhole principle, to guarantee that two of these $n-2$ substrings will be identical, we can take the smallest n such that $n-2 > 8 \Longleftrightarrow n > 10$.

Answer: $\boxed{n = 11}$

6. Let the functions $f : \mathbf{Z} \to \mathbf{Z}$ and $g : \mathbf{Z} \to \mathbf{Z}$ be given by $f(x) = x^2 + 1$ and $g(x) = x^5$. Compute $(g \circ f)(x)$ and expand it as a sum of powers of x with numerical coefficients, using Pascal's triangle.

$(g \circ f)(x) = g(f(x)) = g(x^2+1) = (x^2+1)^5 = (x^2)^5 + \binom{5}{1}(x^2)^4 + \binom{5}{2}(x^2)^3 + \binom{5}{3}(x^2)^2 + \binom{5}{4}x^2 + \binom{5}{5}$ (Binomial formula).

The coefficients $\binom{5}{k}$ can be found from Pascal's triangle:

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      1
     1 1
    1 2 1
   1 3 3 1
  1 4 6 4 1
1 5 10 10 5 1
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So the answer is

$$\boxed{x^{10} + 5x^8 + 10x^6 + 10x^4 + 5x^2 + 1}$$

7. Prove the following by induction: $1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \geq \frac{3}{2} - \frac{1}{n+1}$ for all $n \geq 1$.

Basis: $n = 1$. $1 \overset{?}{\geq} \frac{3}{2} - \frac{1}{2} = 1$. True!

Inductive step: assume $1 + \frac{1}{2^2} + \cdots + \frac{1}{k^2} \geq \frac{3}{2} - \frac{1}{k+1}$ for some $k \geq 1$.

We need to show: $\left(1 + \frac{1}{2^2} + \cdots + \frac{1}{k^2}\right) + \frac{1}{(k+1)^2} \overset{?}{\geq} \frac{3}{2} - \frac{1}{k+2}$.

By the assumption, the LHS is $\geq \frac{3}{2} - \frac{1}{k+1} + \frac{1}{(k+1)^2}$. Thus it suffices to show that

$$\frac{3}{2} - \frac{1}{k+1} + \frac{1}{(k+1)^2} \overset{?}{\geq} \frac{3}{2} - \frac{1}{k+2} \Longleftrightarrow \frac{1}{(k+1)^2} \overset{?}{\geq} \frac{1}{k+1} - \frac{1}{k+2} \Longleftrightarrow$$

$$\frac{1}{(k+1)^2} \overset{?}{\geq} \frac{(k+2)-(k+1)}{(k+1)(k+2)} = \frac{1}{(k+1)(k+2)} \Longleftrightarrow k+1 \overset{?}{\leq} k+2. \text{ True!}$$

8. Solve the recurrence relation $a_n = 3a_{n-1} + 5$ with the initial condition $a_0 = -2$.

Look for the solution in the form $a_n = 3^n \cdot c_1 + c_2$.

For $n=0$: $-2 = a_0 = 3^0 \cdot c_1 + c_2 = c_1 + c_2$.

For $n=1$: $a_1 = 3a_0 + 5 = 3 \cdot (-2) + 5 = -1 = 3c_1 + c_2$.

It follows that $2c_1 = -1 - (-2) = 1 \Longrightarrow c_1 = \frac{1}{2}$,

and $c_2 = -2 - c_1 = -2 - \frac{1}{2} = -\frac{5}{2}$.

Answer: $\boxed{a_n = \frac{3^n - 5}{2}}$

9. Solve the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$, with the initial conditions $a_0 = 1$, $a_1 = 2$.

Try $a_n = r^n$, so we must have $r^n = 6r^{n-1} - 9r^{n-2} \iff$ $r^2 - 6r + 9 = 0 \iff (r-3)^2 = 0$. There is one root $r = 3$.

General solution: $a_n = 3^n c_1 + n \cdot 3^n \cdot c_2$

$n = 0$: $1 = c_1$
$n = 1$: $2 = 3c_1 + 3c_2$

$c_1 = 1$
$c_2 = (2 - 3c_1)/3 = -\frac{1}{3}$.

Answer: $a_n = 3^n - n \cdot 3^n \cdot \frac{1}{3} = 3^{n-1}(3-n)$

10. Using Euclid's Algorithm, find integers x and y such that $29x + 24y = 1$.

Euclid's algorithm for finding gcd(29,24):

$29 = 1 \cdot 24 + 5$
$24 = 4 \cdot 5 + 4$
$5 = 1 \cdot 4 + \boxed{1}$
$4 = 4 \cdot 1 + 0$

gcd is the last non-zero remainder.

So: $1 = 5 - 1 \cdot 4$
$4 = 24 - 4 \cdot 5$
$5 = 29 - 1 \cdot 24$

We have $1 = 5 - 1 \cdot 4 = 5 - (24 - 4 \cdot 5) = -24 + 5 \cdot 5$
$= -24 + 5 \cdot (29 - 1 \cdot 24) = 5 \cdot 29 - 6 \cdot 24$,

Answer: $x = 5$, $y = -6$