Name: Solutions
(PLEASE PRINT)

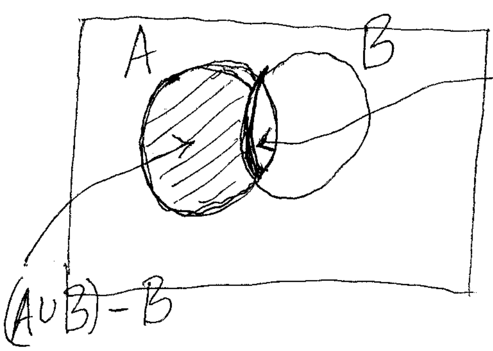
MTH U230 Discrete Mathematics, Fall 2008, Midterm Test

1) Determine whether the compound propositions $\underbrace{(p \wedge \neg q) \vee ((\neg p) \wedge q)}_{r}$ and $p \vee q$ are logically equivalent.

p	q	$\neg p$	$\neg q$	$p \wedge \neg q$	$(\neg p) \wedge q$	r	$p \vee q$
T	T	F	F	F	F	F	T
T	F	F	T	T	F	T	T
F	T	T	F	F	T	T	T
F	F	T	T	F	F	F	F

The columns are not identical, so the propositions are not logically equivalent. (Note: LHS = XOR, while RHS = inclusive OR).

2) What can you say about the sets A and B if we know that $(A \cup B) - B = A$?



$\boxed{A \cap B = \emptyset, \text{ so } A \text{ and } B \text{ are disjoint.}}$

3) Give an example of a bijection (= one-to-one correspondence) from the set $\mathbf{Z}$ of all integers to the set of all odd positive integers $\{1, 3, 5, \ldots\}$.

$f(0)=1,\ f(1)=5,\ f(2)=9,\ \ldots$
$f(-1)=3,\ f(-2)=7,\ f(-3)=11,\ \ldots$

$$\boxed{f(n) = \begin{cases} 4n+1 & \text{if } n \geq 0 \\ -4n-1 & \text{if } n < 0 \end{cases}}$$

4) Show the following identity for Boolean expressions: $(x + \bar{y})(x + \bar{z})(y + z) = x(y + z)$.

$$\underbrace{(x+\bar{y})(x+\bar{z})}(y+z) \overset{\text{"second distributivity}}{=} (x+\bar{y}\bar{z})(y+z) = x(y+z) + \underbrace{\bar{y}y}_{0}\bar{z} + \bar{y}\underbrace{\bar{z}z}_{0} = x(y+z).$$

$P(n)$

5) Show that $\sum_{k=0}^{n}(k+2)2^k = (n+1)2^{n+1}$ for any nonnegative integer n. Induction!

Basis: $n=0$.

$2\cdot 2^0 \stackrel{?}{=} 1\cdot 2^1$ ✓

True!

Inductive step: assume $P(n)$ true for some $n\geq 0$.

Need: $P(n+1)$: $\sum_{k=0}^{n+1}(k+2)\cdot 2^k \stackrel{?}{=} (n+2)\cdot 2^{n+2}$.

The LHS of $P(n+1)$ is $\underbrace{\sum_{k=0}^{n}(k+2)\cdot 2^k}_{} + (n+3)\cdot 2^{n+1} \stackrel{\text{by assumption}}{=} (n+1)\cdot 2^{n+1} + (n+3)\cdot 2^{n+1} = (2n+4)\cdot 2^{n+1} = 2(n+2)\cdot 2^{n+1} = (n+2)\cdot 2^{n+2} = $ RHS of $P(n+1)$. ✓

6) How many social security numbers (that is, 9-digit strings) contain exactly four 0s and no other repeated digits?

non-zero digits

remaining 5 places

$$\binom{9}{4}\cdot P(9,5) = \boxed{\frac{9\cdot 8\cdot 7\cdot 6}{4\cdot 3\cdot 2}\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5}$$

choices for locations of 0s

7) What is the coefficient of x^6y^9 in the expansion of $(2x-5y)^{15}$?

x^6y^9 appears in $\binom{15}{9}\cdot(2x)^6\cdot(-5y)^9$, so the coefficient is

$$\binom{15}{9}\cdot 2^6\cdot(-5)^9 = -\binom{15}{6}\cdot 2^6\cdot 5^9 = \boxed{-\frac{15\cdot 14\cdot 13\cdot 12\cdot 11\cdot 10}{6\cdot 5\cdot 4\cdot 3\cdot 2}\cdot 2^6\cdot 5^9}$$

symmetry of binomial coefficients

8) Let $f_0, f_1, \ldots$ be the Fibonacci sequence defined by $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. Show that $\sum_{k=1}^{n} f_{2k-1} = f_{2n}$ for all $n \geq 1$. Induction.

Basis: $n=1$.

$f_1 \stackrel{?}{=} f_2$: True,

because $f_2 = f_1 + f_0 = 1 = f_1$

Inductive step: assume $\sum_{k=1}^{n} f_{2k-1} = f_{2n}$ for some $n\geq 1$.

Need to show: $\sum_{k=1}^{n+1} f_{2k-1} \stackrel{?}{=} f_{2n+2}$.

LHS $= \underbrace{\sum_{k=1}^{n} f_{2k-1}}_{} + f_{2n+1} \stackrel{\text{by assumption}}{=} \underbrace{f_{2n}}_{} + f_{2n+1} \stackrel{\text{by definition of Fibonacci numbers}}{=} f_{2n+2} = $ RHS. ✓