# REMARKABLE RECURRENCES (PRISM, MAY 24-27, 2010)

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This four-day research experience for undergraduates explores some algebraic recurrence relations which attracted a lot of interest recently because of their unexpected appearance in various areas of mathematics and theoretical physics. Starting with classical families of Fibonacci, Pell and Markov numbers, we move on to much more recent Somos sequences and other recurrences of similar kind. We finish by exploring a general mechanism for producing "interesting" systems of algebraic recurrences, based on the new operation of *quiver mutation*.

## 1. Fibonacci numbers

As a warm-up, we start with famous *Fibonacci numbers* $0, 1, 1, 2, 3, 5, 8, 13, 21, ...$ given by the initial conditions

$$f_0 = 0, \quad f_1 = 1$$

and the recurrence relation

$$f_n = f_{n-1} + f_{n-2} \quad (n \geq 2).$$

They appear in a lot of counting problems.

**Example.** How many ways are there to express a positive integer $n$ as the sum of 1's and 2's (the order of terms matters)? For instance, for $n = 3$ there are three such presentations:

$$3 = 2 + 1 = 1 + 2 = 1 + 1 + 1.$$

**Answer:** $f_{n+1}$. **Proof:** denoting the number in question by $c_n$, note that $c_1 = 1 = f_2, c_2 = 2 = f_3$, and that these numbers satisfy the Fibonacci recurrence relation (to see this, count separately the expressions starting with 1 and with 2).

**Problem 1.1.** How many ways are there to tile $2 \times n$ checker-board with $1 \times 2$ dominos?

**Problem 1.2.** How many bit strings of length $n$ (that is, strings consisting of 0's and 1's) have no two consecutive 1's?

**Problem 1.3.** Consider the sequence $(g_n)$ formed by taking every second Fibonacci number: $1, 2, 5, 13, 34, 89, ....$ Thus, we have $g_n = f_{2n+1}$. Show that these numbers satisfy the recurrence relation $g_n = 3g_{n-1} - g_{n-2}$.

**Problem 1.4.** How fast do the Fibonacci numbers grow? Compute the ratio $f_n/f_{n-1}$ for $n = 2, \ldots, 10$. Can you guess what is the limit of these ratios as $n$ goes to infinity?

**Problem 1.5.** Find all numbers $r$ such that the sequence of powers $1, r, r^2, r^3, \ldots$ satisfies the Fibonacci recurrence.

**Problem 1.6.** Show that the Fibonacci numbers are given by the formula

$$f_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right) .$$

The number $\varphi = \frac{1+\sqrt{5}}{2}$ is known as the *golden number* or the *golden ratio.* As a consequence of Problem 1.6, the Fibonacci number $f_n$ is the closest integer to $\varphi^n/\sqrt{5}$.

## 2. Pell pairs

By a *Pell pair* we mean a pair $(x, y)$ of positive integers satisfying the equation

$$|x^2 - 2y^2| = 1 .$$

The simplest Pell pair is $(1, 1)$. Can you produce more Pell pairs?

Note that the equation $x^2 - 2y^2 = 0$ has no solutions in positive integers: such a solution would satisfy $x/y = \sqrt{2}$, which is impossible since $\sqrt{2}$ is irrational. The ratios $x/y$ of the two components in Pell pairs can be seen as "best possible" rational approximations to $\sqrt{2}$.

To find all Pell pairs, we will use the following remarkable fact.

**Problem 2.1.** Show that if $(x, y)$ is a Pell pair, then so is $(x + 2y, x + y)$.

This fact allows us to define a sequence $((x_n, y_n))$ of Pell pairs starting with $(x_1, y_1) = (1, 1)$ and satisfying the recurrence

$$(x_n, y_n) = (x_{n-1} + 2y_{n-1}, x_{n-1} + y_{n-1}) \quad (n \geq 2) .$$

Thus, the sequence starts with $(1, 1), (3, 2), (7, 5), ....$

**Problem 2.2.*** Show that this sequence contains *all* Pell pairs. [**Hint:** find and use the formulas that express $(x_{n-1}, y_{n-1})$ in terms of $(x_n, y_n)$.]

**Problem 2.3.** Show that each of the sequences $(x_n)$ and $(y_n)$ satisfies the same recurrence relation:

$$x_n = 2x_{n-1} + x_{n-2}, \quad y_n = 2y_{n-1} + y_{n-2} \quad (n \geq 3) .$$

**Problem 2.4.** Consider the sequence $z_0, z_1, \ldots$ given by $z_n = y_{2n+1}$. Show that it can be defined by the initial conditions $z_0 = 1, z_1 = 5$, and the recurrence relation $z_n = 6z_{n-1} - z_{n-2}$ for $n \geq 2$.

**Problem 2.5.** Show that each Pell pair $(x_n, y_n)$ is given by the formulas

$$x_n = \frac{1}{2}((1+\sqrt{2})^n + (1-\sqrt{2})^n), \quad y_n = \frac{1}{2\sqrt{2}}((1+\sqrt{2})^n - (1-\sqrt{2})^n) .$$

We finish this section with the following open-ended problem.

**Problem 2.6.*** State and solve the analogs of the above problems for the pairs of positive integers $(x, y)$ satisfying the equation $|x^2 - xy - y^2| = 1$. What would be an appropriate name for these pairs?

## 3. Markov triples

A *Markov triple* (aka *Markoff triple*) is a triple of positive integers $(x, y, z)$ satisfying the *Markov equation*

$$x^2 + y^2 + z^2 = 3xyz\ .$$

The simplest Markov triple is $(1, 1, 1)$. Can you find some others?

**Problem 3.1.** Show that if $(x, y, z)$ is a Markov triple then

$$\frac{y^2 + z^2}{x} = 3yz - x\ .$$

**Problem 3.2.** Show that if $(x, y, z)$ is a Markov triple then so is $(\frac{y^2+z^2}{x}, y, z)$.

Since the Markov equation is symmetric with respect to $x, y, z$, there are the following three transformations of Markov triples which we refer to as *mutations*:

$$\mu_1 : (x, y, z) \to (\frac{y^2 + z^2}{x}, y, z), \quad \mu_2 : (x, y, z) \to (x, \frac{x^2 + z^2}{y}, z),$$

$$\mu_3 : (x, y, z) \to (x, y, \frac{x^2 + y^2}{z}).$$

For instance, applying these mutations to the initial Markov triple $(1, 1, 1)$ we get the three triples $(2, 1, 1), (1, 2, 1)$, and $(1, 1, 2)$. As for the triple $(2, 1, 1)$, the mutation $\mu_1$ brings it back to $(1, 1, 1)$, while $\mu_2$ and $\mu_3$ transform it into $(2, 5, 1)$ and $(2, 1, 5)$, respectively.

**Problem 3.3.*** Show that every Markov triple can be obtained by applying a finite sequence of mutations to the initial triple $(1, 1, 1)$.

Note that every mutation applied twice returns any Markov triple to itself. Thus we can depict all Markov triples by placing them into the vertices of an infinite trivalent tree, whose edges are labeled by $1, 2$, or $3$ in such a way that at every vertex three incoming edges have three different labels. To do this, place $(1, 1, 1)$ into some vertex, and start producing new triples by moving away from it in the following way: each time you move along an edge labeled, say by 1, you apply the mutation $\mu_1$.

**Problem 3.4.** Let $(g_0, g_1, g_2, \dots) = (1, 2, 5, 13, 34, 89, ...)$ be the sequence of alternate Fibonacci numbers from Problem 1.3. Show that $(1, g_{n-1}, g_n)$ is a Markov triple for any $n \geq 1$.

**Problem 3.5.** Let $(z_0, z_1, z_2, \dots) = (1, 2, 5, 12, 29, ...)$ be the sequence of alternate "Pell denominators" from Problem 2.4. Show that $(2, z_{n-1}, z_n)$ is a Markov triple for any $n \geq 1$.

## 4. Somos-like sequences

Here are several interesting sequences given by initial conditions and recurrence relations:

$$(S2):\ x_1 = x_2 = 1, \quad x_n = \frac{x_{n-1}^2 + 1}{x_{n-2}} \quad (n \geq 3);$$

$$(S3):\ x_1 = x_2 = x_3 = 1, \quad x_n = \frac{x_{n-1}x_{n-2} + 1}{x_{n-3}} \quad (n \geq 4);$$

$$(S4):\ x_1 = \cdots = x_4 = 1, \quad x_n = \frac{x_{n-1}x_{n-3} + x_{n-2}^2}{x_{n-4}} \quad (n \geq 5);$$

$$(S5):\ x_1 = \cdots = x_5 = 1, \quad x_n = \frac{x_{n-1}x_{n-4} + x_{n-2}x_{n-3}}{x_{n-5}} \quad (n \geq 6).$$

The sequences $(S4)$ and $(S5)$ are called *Somos-4* and *Somos-5*, respectively. They were discovered by M. Somos around 1990.

The common surprising feature of all these sequences is that all their terms turn out to be *integers*. This is surprising because the recurrence relations involve *division*.

For the sequence $(S2)$, an explanation can be given as follows.

**Problem 4.1.** Show that the $(S2)$-sequence $(x_2, x_3, \dots)$ is the sequence of alternate Fibonacci numbers which we already encountered in Problems 1.3 and 3.3.

For the sequences $(S3)-(S5)$, the fact that the terms are integers, is more difficult to explain. As often happens in mathematics, it turns out to be simpler to prove a *stronger* property. Namely, think of the initial terms of each sequence as independent variables; then the rest of the terms can be viewed as rational functions in these variables. For instance, for $(S2)$, we think of $x_1$ and $x_2$ as independent variables; then we have

$$x_3 = \frac{x_2^2 + 1}{x_1}, \quad x_4 = \frac{x_3^2 + 1}{x_2} = \frac{(x_2^2 + 1)^2 + x_1^2}{x_1^2 x^2},$$

$$x_5 = \frac{x_4^2 + 1}{x_3} = \frac{((x_2^2 + 1)^2 + x_1^2)^2 + x_1^4 x_2^2}{x_1^3 x_2^2 (x_2^2 + 1)} = \frac{(x_2^2 + 1)^3 + 2x_1^2(x_2^2 + 1) + x_1^4}{x_1^3 x_2^2}.$$

**Problem 4.2.** For each of the sequences $(Sr)$ with $r = 3, 4, 5$, compute (with the help of a calculator or computer, if you wish) the terms $x_{r+1}, \dots, x_{2r+1}$ as rational functions in $x_1, \dots, x_r$.

**Problem 4.3.** Let $\mu_1, \mu_2$, and $\mu_3$ be the mutations introduced in Section 4. Thus, they are given by

$$\mu_1 : (x, y, z) \to (\frac{y^2 + z^2}{x}, y, z), \quad \mu_2 : (x, y, z) \to (x, \frac{x^2 + z^2}{y}, z),$$

$$\mu_3 : (x, y, z) \to (x, y, \frac{x^2 + y^2}{z}).$$

Think of the initial triple $(x, y, z)$ as a triple of independent variables and compute the triples obtained from it by applying the mutations $\mu_1$, $\mu_2$, and $\mu_1$ in this order.

## 5. Laurent phenomenon: variations of $(S2)$

Recall some basic algebraic terminology. An (integer) *polynomial* in variables $x_1, \dots, x_n$ is a sum of terms of the form $cx_1^{a_1} \cdots x_n^{a_n}$ with $c$ an integer, and $a_1, \dots, a_n$ nonnegative integers. A *rational function* is a ratio of two polynomials. Among rational functions, the simplest are *Laurent polynomials* – those whose denominator is just a monomial.

The computations in the problems from Section 4 suggest that for each sequence $(Sr)$, every term $x_n$ turns out to be a Laurent polynomial with integer coefficients in the initial terms $x_1, \dots, x_r$, that is, a polynomial with integer coefficients divided by a monomial. This of course implies that if all the initial terms are specialized to 1, then each $x_n$ becomes an integer.

This unexpected appearance of Laurent polynomials is referred to as the *Laurent phenomenon.* To test it in more cases, we consider some variations of the sequence $(S2)$.

**Problem 5.1.** Show that the sequence given by the recurrence $x_n = \frac{x_{n-1}+1}{x_{n-2}}$ exhibits the Laurent phenomenon. Show that this sequence is periodical. What is its period?

**Problem 5.2.** Consider the sequence given by the recurrence $x_n = \frac{x_{n-1}^b+1}{x_{n-2}}$, where $b$ is some fixed positive integer. Show that $x_3, x_4$ and $x_5$ are Laurent polynomials in $x_1$ and $x_2$.

**Problem 5.3.** Fix positive integers $b$ and $c$, and consider the sequence given by the recurrence

$$x_n = \begin{cases} \frac{x_{n-1}^b+1}{x_{n-2}} & \text{if } n \text{ is odd;} \\ \frac{x_{n-1}^c+1}{x_{n-2}} & \text{if } n \text{ is even.} \end{cases}$$

Show that $x_3, x_4$ and $x_5$ are Laurent polynomials in $x_1$ and $x_2$.

**Problem 5.4.** Show that the sequence in Problem 5.3 exhibits the Laurent phenomenon if $(b, c) = (1, 2)$ or $(b, c) = (1, 3)$. Show that both these sequences are periodical. What are their periods?

We finish the section with an open-ended problem.

**Problem 5.5.** Explore some other sequences of a similar kind. For instance, what about the sequence, where the exponents of $x_{n-1}$ in consecutive recurrence relations form a sequence $1, 1, 2, 1, 1, 2, \ldots$? Does it exhibit the Laurent phenomenon?

## 6. Quiver mutations

A large class of recurrences exhibiting the Laurent phenomenon was found and explored in the paper "The Laurent Phenomenon" by S. Fomin and A. Zelevinsky published in "Advances in Applied Mathematics." This class is described in terms of *quivers* and their *mutations.*

By a *quiver* we mean a finite directed graph, that is, a finite collection of *vertices*, some of which are joined by *arrows.* We assume that vertices are labeled with integers $1, \ldots, n$. We do not allow loops (i.e., arrows with the same tail and head), and also pairs of arrows forming an oriented 2-cycle. But we do allow multiple arrows: it is possible that there are say 5 arrows from a vertex $j$ to a vertex $i$, in which case we say that an arrow $j \to i$ has *multiplicity* 5.

For every vertex $k$ of a quiver $Q$, the *mutation at* $k$ is defined as an operation (denoted $\mu_k$) that transforms $Q$ into a new quiver $Q'$ on the same vertices, that is obtained from $Q$ by the following three-step procedure:

**Step 1.** For every incoming arrow $j \to k$ of some multiplicity $a$ at a chosen vertex $k$, and every outgoing arrow $k \to i$ of multiplicity $b$, create a "composite" arrow $j \to i$ of multiplicity $ab$.

**Step 2.** Reverse all arrows at $k$.

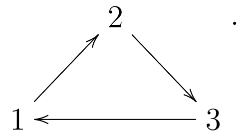
**Step 3.** Remove all oriented 2-cycles that could appear as a result of creating new arrows in Step 1).

**Problem 6.1.** Show that if you apply the mutation $\mu_k$ twice, then you return to the initial quiver $Q$.

The quiver mutation is implemented as a Java applet by B. Keller, and is available online at `http://www.math.jussieu.fr/~keller/quivermutation/`. It has plenty of interesting properties which can be explored with the help of this applet.

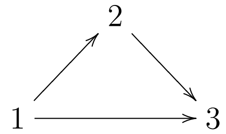
Here are a few sample problems.

**Problem 6.2.** Show that a cyclically oriented triangle



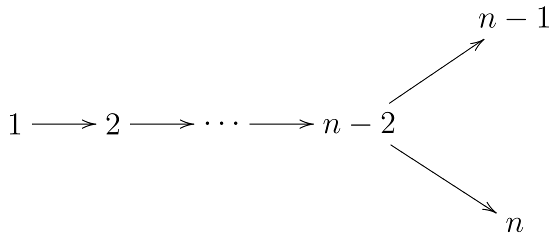

can be mutated into a chain $1 \longrightarrow 2 \longleftarrow 3$.

**Problem 6.3.** Can the triangle



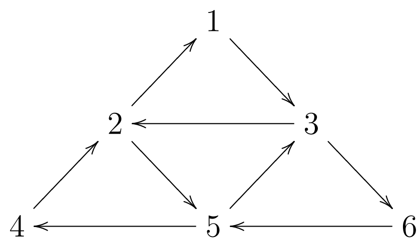

be mutated into an oriented tree?

**Problem 6.4.** Show that, for any $n \geq 4$, a cyclically oriented $n$-cycle can be mutated into a quiver




**Problem 6.5.** Show that every non-cyclically oriented $n$-cycle can be mutated into a quiver with a multiple arrow.

**Problem 6.6.** Can you mutate the quiver




into an oriented tree?

## 7. More on quiver mutations

A vertex of a quiver is called a *source* (resp. a *sink*) if it has no incoming (resp. outgoing) arrows. A mutation at a source or a sink is called *shape-preserving*: it just amounts to reversing all arrows at the vertex.

**Problem 7.1.** Show that any two orientations of a chain with $n$ vertices can be obtained from each other by mutations.

**Problem 7.2.*** More generally, show that any two orientations of a tree can be obtained from each other by mutations.

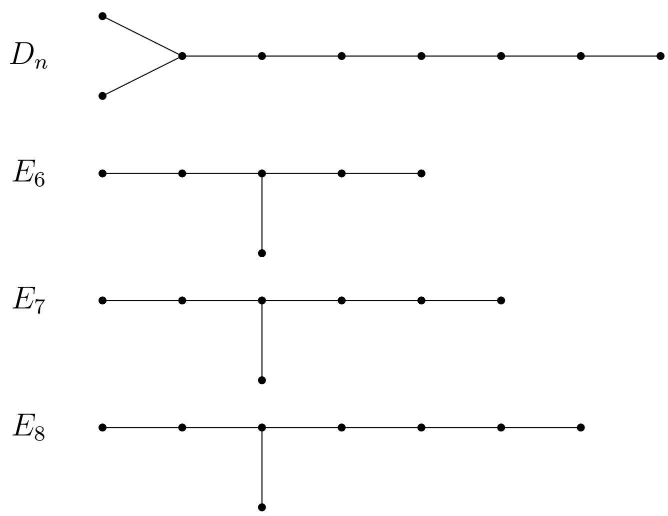
We say that a quiver is

- *of finite (mutation) type* if any quiver obtained from it by mutations (including itself) has no multiple edges;
- *(mutation) tame* if it is not of finite type but there are only finitely many quivers it can be mutated into;
- *(mutation) wild* if it is neither of finite type, nor tame.

**Problem 7.3.** Show that a cyclically oriented triangle is of finite type, a non-cyclically oriented triangle is tame, and the quiver $1 \xrightarrow{3} 2 \longrightarrow 3$ is wild.

A classification of quivers of finite type was obtained by S. Fomin and A. Zelevinsky in 2003. They proved the following.

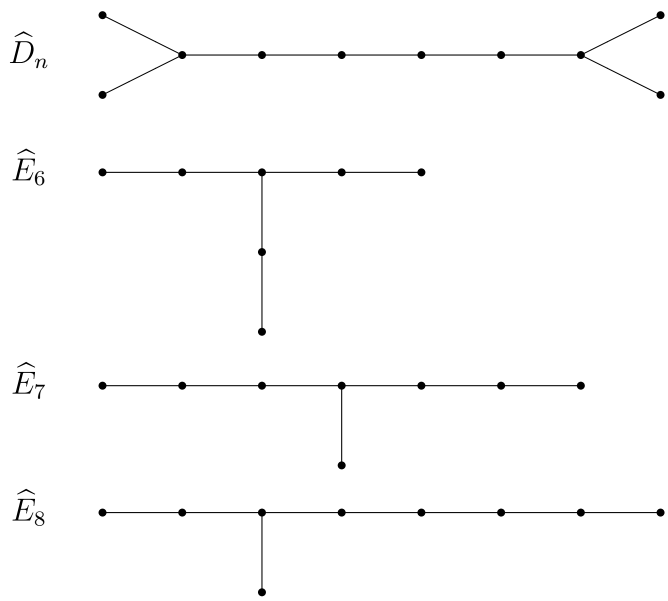
**Theorem.** A quiver is of finite type if and only if it can be mutated into an orientation of a tree from the following list:




These quivers are called *Dynkin quivers*.

We call a quiver an *extended Dynkin quiver* if it is an oriented tree which is not a Dynkin quiver but becomes a disjoint union of Dynkin quivers after removing any vertex.

**Problem 7.4.*** Show that any extended Dynkin quiver is an orientation of a tree from the following list:




**Problem 7.5.** Show that each of the extended Dynkin quivers in Problem 7.4 is not of finite type, that is, it can be mutated into a quiver with a multiple arrow. [In fact, all these quivers are tame but this is much harder to show.]

## 8. Quiver mutations and the Laurent phenomenon

In this concluding section we discuss how to use quiver mutations for producing a lot of algebraic recurrences that exhibit the Laurent phenomenon. Given a quiver on $n$ vertices, we can attach an independent variable $x_k$ to every vertex $k = 1, \ldots, n$. Thus we obtain an $n$-tuple of independent variables $x_1, \ldots, x_n$ that we refer to as an *initial cluster.*

We accompany a quiver mutation $\mu_k$ at a given vertex $k$ with the following transformation of the initial cluster: replace the variable $x_k$ with a rational function $x'_k$ in $x_1, \ldots, x_n$ defined as follows. Suppose the incoming arrows at $k$ are from vertices $j_1, \ldots j_p$, with multiplicities $a_1, \ldots, a_p$, and the outgoing arrows at $k$ are to vertices $i_1, \ldots, i_q$, with multiplicities $b_1, \ldots, b_q$. Then we set

$$x'_k = \frac{x_{j_1}^{a_1} \cdots x_{j_p}^{a_p} + x_{i_1}^{b_1} \cdots x_{i_q}^{b_q}}{x_k}$$

(with the convention that an empty product is understood as 1).

This way, we view a mutation $\mu_k$ as an operation transforming a quiver $Q$ together with an attached cluster $\mathbf{x} = \{x_1, \ldots, x_n\}$ into a new quiver $Q'$ together with a new cluster $\mathbf{x}' = \{x_1, \ldots, x'_k, \ldots, x_n\}$. Then we can apply a mutation at an arbitrary vertex to a pair $(Q', \mathbf{x}')$ and continue this process by mutating at an arbitrary sequence of vertices. The elements of all the clusters obtained in this way are called *cluster variables*. Thus, every cluster variable by construction will be a rational function in the initial cluster variables $x_1, \ldots, x_n$. Now we are ready to state the "Laurent phenomenon" result mentioned in the beginning of Section 6.

**Theorem.** For every initial quiver, all the cluster variables obtained by repeated mutations are Laurent polynomials in the initial cluster variables $x_1, \ldots, x_n$.

It turns out that (almost) all instances of the Laurent phenomenon observed earlier can be seen as special instances of this theorem. For instance, applying to the quiver $1 \longrightarrow 2$ the mutations $\mu_1, \mu_2, \mu_1, \mu_2, \ldots$, we obtain the sequence from Problem 5.1.

**Problem 8.1.** Show that the sequence of mutations $\mu_1, \mu_2, \mu_1, \mu_2, \ldots$ applied to the quiver with two vertices and one double arrow, produces a sequence $(S2)$ from Section 4.

**Problem 8.2.** Show that the Laurent phenomenon for Markov triples can be obtained by applying the above theorem to the cyclically oriented triangle, where each arrow has multiplicity 2.

**Problem 8.3.** Show that the sequence of mutations $\mu_1, \mu_2, \mu_3, \mu_1, \mu_2, \mu_3, \ldots$ applied to the quiver in Problem 6.3, produces a sequence $(S3)$ from Section 4.

**Problem 8.4.** Construct a quiver on four vertices such that the sequence $(S4)$ from Section 4 is obtained from it by applying repeatedly the series of mutations $\mu_1, \mu_2, \mu_3, \mu_4$ (in this order).

**Problem 8.5.** Construct a quiver on five vertices such that the sequence $(S5)$ from Section 4 is obtained from it by applying repeatedly the series of mutations $\mu_1, \mu_2, \mu_3, \mu_4, \mu_5$ (in this order).