Name: Solutions

(PLEASE PRINT)

MTH U165 Intro to Mathematical Reasoning, Fall 2007, Quiz 7

1) Prove by induction that $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\cdots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$ for any positive integer n.

Basis: $n=1$

$\frac{1}{1\cdot 3} \overset{?}{=} \frac{1}{2\cdot 1+1}$

$\frac{1}{3}=\frac{1}{3}$: True

Inductive step: assume $\frac{1}{1\cdot 3}+\cdots+\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1}$ for some $k\geq 1$.

Need to show: $\frac{1}{1\cdot 3}+\cdots+\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2k+1)(2k+3)} \overset{?}{=} \frac{k+1}{2k+3}$.

$\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)} \overset{?}{=} \frac{k+1}{2k+3} \Longleftrightarrow k(2k+3)+1 \overset{?}{=} (k+1)(2k+1)$

$2k^2+3k+1 \overset{?}{=} 2k^2+2k+k+1$: True!

2) Solve each of the following recurrence relations by giving an explicit formula for a_n. In each case check your answer by computing a_2 directly from the recurrence relation and from your formula.

(a) $a_n = 3a_{n-1}-2$, with the initial condition $a_0=3$.

$a_n = c_1\cdot 3^n + c_2$

$a_0 = 3 = c_1 + c_2$

$a_1 = 3a_0 - 2 = 7 = 3c_1 + c_2$

$2c_1 = 4$

$c_1 = 2$

$c_2 = 3 - c_1 = 1$

Answer: $a_n = 2\cdot 3^n + 1$

Check: $a_2 = 3a_1 - 2 = 19$

$a_2 = 2\cdot 9 + 1 = 19$ //

(b) $a_n = 6a_{n-1}-9a_{n-2}$, with the initial conditions $a_0=1$, $a_1=9$.

Characteristic equation:

$r^2 = 6r - 9 \Longleftrightarrow r^2 - 6r + 9 = 0$

$\Longleftrightarrow (r-3)^2 = 0$

The only root: $\boxed{r=3}$

$a_n = c_1\cdot 3^n + c_2\cdot n\cdot 3^n$

$a_0 = \boxed{1 = c_1}$

$a_1 = 9 = 3c_1 + 3c_2$

$3 = c_1 + c_2$

$\boxed{c_2 = 2}$

Answer:

$a_n = 3^n(2n+1)$

Check:

$a_2 = 6\cdot 9 - 9 = 45$

$\parallel$ ✓

$9\cdot 5$.

(c) $a_n = 3a_{n-1}+10a_{n-2}$, with the initial conditions $a_0=7$, $a_1=0$.

Char. equation: $r^2 = 3r + 10 \Longleftrightarrow$

$r^2 - 3r - 10 = 0 \Longleftrightarrow (r+2)(r-5) = 0$

$r_1 = 5$, $r_2 = -2$,

$a_n = c_1 5^n + c_2(-2)^n$

$a_0 = 7 = c_1 + c_2$

$a_1 = 0 = 5c_1 - 2c_2$

$c_1 = 2$

$c_2 = 5$

Answer: $\boxed{a_n = 2\cdot 5^n + 5\cdot(-2)^n}$

Check: $a_2 = 70$

// ✓

$2\cdot 5^2 + 5\cdot 4$