

# COBORDISM INVARIANCE OF THE INDEX OF A TRANSVERSALLY ELLIPTIC OPERATOR

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## 1. THE $\text{Spin}^{\mathbb{C}}$ -DIRAC OPERATOR AND THE $\text{Spin}^{\mathbb{C}}$ -QUANTIZATION

In this section we reformulate the “ $\text{Spin}^{\mathbb{C}}$ -quantization commutes with cobordism” principal in a more analytic language. In Subsection 1.1, we briefly recall the notion of  $\text{Spin}^{\mathbb{C}}$ -Dirac operator. We refer the reader to [2, 6] for details. We also express the  $\text{Spin}^{\mathbb{C}}$ -quantization of an orbifold  $M = X/G$  in terms of the index of the lift of the  $\text{Spin}^{\mathbb{C}}$ -Dirac operator to  $X$ . In Subsection 1.2, we explain the relationship between such lifts associated with cobordant  $\text{Spin}^{\mathbb{C}}$ -structures. This leads us to a notion of a cobordism between transversally elliptic operators. To show that  $\text{Spin}^{\mathbb{C}}$ -quantization of orbifolds commutes with cobordism it is then enough to prove that the index of transversally elliptic operators commutes with cobordisms, which will be shown in the subsequent sections.

**1.1. A Dirac operator associated with a  $\text{Spin}^{\mathbb{C}}$ -structure.** Suppose  $M$  is a compact oriented  $m$ -dimensional orbifold. Let  $(P, p)$  be a  $\text{Spin}^{\mathbb{C}}$ -structure on  $M$ . Recall from Section D.2 that here  $P$  is a principal  $\text{Spin}^{\mathbb{C}}(n)$ -bundle over  $M$  and  $p : P \rightarrow \text{GL}_+(TM)$  is a  $\text{Spin}^{\mathbb{C}}(n)$ -equivariant map, which gives rise to an  $\text{SO}(n)$ -structure on  $TM$  and, thus, to a Riemannian metric  $g^M$  on  $M$ .

The group  $\text{Spin}^{\mathbb{C}}(n)$  has a canonical unitary representation  $S$ , called the *space of spinors*. The Hermitian orbibundle  $\mathcal{S} = P \times_{\text{Spin}^{\mathbb{C}}(n)} S$  is called the *spinor bundle* over  $M$ .

**Example.** If the  $\text{Spin}^{\mathbb{C}}$ -structure on  $M$  is given by an almost complex structure and a complex line bundle  $\mathbb{L}$  (cf. Section D.5) then the spinor bundle is isomorphic to  $\Lambda^{\bullet}(T^{0,1}M)^* \otimes \mathbb{L}$ , where  $\Lambda^{\bullet}(T^{0,1}M)^*$  is the bundle of anti-holomorphic forms on  $M$ .

There is a canonical action  $c : TM \rightarrow \text{End}(\mathcal{S})$  of the tangent bundle  $TM$  on  $\mathcal{S}$  by skew-adjoint endomorphisms, such that  $c(v)^2 = -|v|^2$  where  $|v|$  denotes the norm of the vector  $v \in TM$  with respect to the Riemannian metric  $g^M$ .

Let  $\nabla^{\mathcal{S}}$  be a Hermitian connection on  $\mathcal{S}$ . The  $\text{Spin}^{\mathbb{C}}$ -Dirac operator on  $\mathcal{S}$  is the first order differential operator

$$D := \sum_{i=1}^n c(e_i) \nabla_{e_i}^{\mathcal{S}} : C^{\infty}(M, \mathcal{S}) \rightarrow C^{\infty}(M, \mathcal{S}), \quad (1)$$

where  $e_1, \dots, e_m$  is an orthonormal frame of  $TM$  (the operator  $D$  is independent of the choice of this frame). If the connection  $\nabla^{\mathcal{S}}$  was properly chosen, which we will henceforth assume, then the operator  $D$  is self-adjoint, cf. [2, Proposition 3.44].

Suppose now that  $m = \dim M$  is even. Then the spinor bundle possesses a natural grading  $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$  such that

$$D : C^{\infty}(M, \mathcal{S}^{\pm}) \rightarrow C^{\infty}(M, \mathcal{S}^{\mp}).$$

We denote by  $D^\pm$  the restriction of  $D$  to the space  $C^\infty(M, \mathcal{S}^\pm)$ . Note that  $(D^+)^* = D^-$ .

By definition, the  $\text{Spin}^\mathbb{C}$ -quantization  $Q(M)$  of  $M$  is equal to the index the operator  $D^+$ :

$$Q(M) := \dim \text{Ker } D^+ - \dim \text{Coker } D^+ = \dim \text{Ker } D^+ - \dim \text{Ker } D^-.$$

Suppose now that  $M = X/G$  is a presentation of the orbifold  $M$ . Here  $X$  is a smooth compact manifold with a locally free action of a compact group  $G$ . Let  $g^X$  be the lift of the Riemannian metric on  $M$  to  $X$ . Then the action of  $G$  on  $X$  preserves  $g^X$ .

Denote by  $\tilde{D}^\pm$  the lift of  $D^\pm$  to  $X$ . Then  $\tilde{D}^\pm$  is a  $G$ -invariant transversally elliptic operator on  $X$  (see Subsection 3.1 for a definition of a transversally elliptic operator). The group  $G$  acts on the kernel of  $\tilde{D}^\pm$  and  $\text{Ker } D^\pm$  is naturally isomorphic to the  $G$ -invariant part  $(\text{Ker } \tilde{D}^\pm)^G$  of  $\text{Ker } \tilde{D}^\pm$ , cf. [7, §1]. Hence,

$$Q(M) = \dim (\text{Ker } \tilde{D}^+)^G - \dim (\text{Ker } \tilde{D}^-)^G.$$

**1.2. A cobordism of Dirac operators.** Let  $M_0 = X_0/G$  and  $M_1 = X_1/G$  be presented orbifolds of the same even dimension  $m = 2k$ , endowed with  $\text{Spin}^\mathbb{C}$ -structures. Assume that there is given a  $\text{Spin}^\mathbb{C}$ -cobordism between  $M_0$  and  $M_1$ . In other words, we assume the following data:

A compact oriented manifold  $W$  with boundary; a locally free action of a compact Lie group  $G$  on  $W$ ; the representation of the boundary of  $W$  as a disjoint union of  $-X_0$  and  $X_1$  (here,  $-X_0$  is the manifold  $X_0$  with the opposite orientation); the  $\text{Spin}^\mathbb{C}$ -structure on  $W/G$ , which, by restriction, induces  $\text{Spin}^\mathbb{C}$ -structures on  $X_i/G$ ,  $i = 0, 1$ ; the isomorphisms between the orbifolds  $M_i$  and  $X_i/G$ ,  $i = 0, 1$ , which carry the  $\text{Spin}^\mathbb{C}$ -structures on  $M_i$  to the  $\text{Spin}^\mathbb{C}$ -structures on  $X_i/G$ .

Recall from the previous subsection, that there is a natural Riemannian metric on  $W$ . Using this metric we can identify the union of the cylinders  $X_0 \times [0, \varepsilon)$  and  $X_1 \times (-\varepsilon, 0]$  with a neighborhood  $U \subset W$  of the boundary of  $W$ . We denote by  $t : U \rightarrow \mathbb{R}$  the projection onto the second factor. By a slight abuse of notation, we denote by the same letter  $t$  the induced map  $U/G \rightarrow \mathbb{R}$ . Let  $dt \in T(U/G) \subset T(W/G)$  be the corresponding vector (we use the Riemannian metric to identify the tangent and the cotangent bundles to  $W/G$ ). Set

$$\gamma := c(dt).$$

Let  $\mathcal{S}, \mathcal{S}_i$  denote the spinor bundles on  $W/G$  and  $M_i = X_i/G$  respectively. Recall from the previous section that there is a natural grading  $\mathcal{S}_i = \mathcal{S}_i^+ \oplus \mathcal{S}_i^-$ . There is a natural isomorphism between the restriction of  $\mathcal{S}$  to  $M_i$  and  $\mathcal{S}_i$ . Under this isomorphism we have  $\gamma|_{\mathcal{S}_i^\pm} = \pm\sqrt{-1}$ .

Fix a connection on  $\mathcal{S}$ . It induces connections on  $\mathcal{S}_i$ . Let  $\tilde{D}, \tilde{D}_i$  denote the lifts of the corresponding Dirac operators to  $W$  and  $X_i$  respectively. From (1), we see that the restriction of  $\tilde{D}$  to the neighborhood of  $X_i$ ,  $i = 0, 1$  has the form

$$\tilde{D} = \gamma \frac{\partial}{\partial t} + \tilde{D}_i, \tag{2}$$

where  $\frac{\partial}{\partial t}$  denotes the covariant derivatives along  $t$ .

Thus to show that the  $\text{Spin}^\mathbb{C}$ -quantizations commutes with cobordism, it is enough to prove that the indexes of the transversally elliptic operators  $\tilde{D}_i$  on  $X_i$ ,  $i = 0, 1$  coincide, whenever there exists a  $G$ -equivariant cobordism  $W$  between  $X_0$  and  $X_1$  and a  $G$ -invariant transversally

elliptic operator  $\tilde{D}$  on  $W$ , whose restriction to a neighborhood of the boundary satisfies (2). This statement will be made more precise and proven below.

## 2. THE SUMMARY OF THE RESULTS

2.1. Let  $X$  be a compact  $n$ -dimensional Riemannian manifold on which a compact Lie group  $G$  acts by isometries. Let  $E^+, E^-$  be  $G$ -equivariant Hermitian vector bundles over  $X$ . Let  $A^+ : C^\infty(X, E^+) \rightarrow C^\infty(X, E^-)$  be a  $G$ -invariant transversally elliptic differential operator of order 1 (cf. [1] or Section 3 of this appendix).

Let  $A^- : C^\infty(X, E^-) \rightarrow C^\infty(X, E^+)$  be the formal adjoint of  $A^+$  and consider the operator

$$A := \begin{bmatrix} 0 & A^- \\ A^+ & 0 \end{bmatrix} : C^\infty(X, E^+ \oplus E^-) \rightarrow C^\infty(X, E^+ \oplus E^-).$$

2.2. **The distributional index.** The kernel  $\text{Ker}(A^\pm) \subset L^2(X, E^\pm)$  is a closed  $G$ -invariant subspace, and, hence, can be considered as unitary representations of  $G$ .

Let us denote by  $\hat{G}$  the set of all equivalence classes of irreducible representations of  $G$ . For  $\rho \in \hat{G}$ , we denote by  $\text{Ker}_\rho(A^\pm) := \text{Hom}_G(\rho, \text{Ker}(A^\pm))$  the  $\rho$ -component of  $\text{Ker}(A^\pm)$ .

Atiyah, [1], showed that, for each  $\rho \in \hat{G}$ , the dimension of the spaces  $\text{Ker}_\rho(A^\pm)$  is finite. Moreover, the formal sum

$$\text{char Ker}(A^\pm) := \sum_{\rho \in \hat{G}} \dim \text{Ker}_\rho(A^\pm) \cdot \text{char } \rho \quad (3)$$

converges to a distribution on  $G$ .

The *distributional index*  $\text{ind}^G(A)$  is defined by

$$\text{ind}^G A := \text{char Ker}(A^+) - \text{char Ker}(A^-) \in \mathcal{D}'(G)^{\text{inv}}, \quad (4)$$

where  $\mathcal{D}'(G)^{\text{inv}}$  denotes the space of distributions on  $G$  invariant under the inner automorphisms of  $G$ . Set  $\text{ind}_\rho^G A := \dim \text{Ker}_\rho(A^+) - \dim \text{Ker}_\rho(A^-)$ . Then

$$\text{ind}^G A = \sum_{\rho \in \hat{G}} \text{ind}_\rho^G(A) \cdot \text{char } \rho. \quad (5)$$

2.3. **The case when  $X$  is a boundary.** Suppose now that  $X$  is a boundary of a compact  $G$ -manifold  $W$  and that  $F$  is a  $G$ -equivariant vector bundle over  $W$ , whose restriction to  $X$  is  $G$ -equivariantly isomorphic to  $E$ . Note, that we do not assume that the bundle  $F$  is graded.

We choose an equivariant identification of a neighborhood  $U$  of the boundary of  $W$  with the product  $X \times (-\varepsilon, 0]$  and we denote by  $t : X \times (-\varepsilon, 0] \rightarrow (-\varepsilon, 0]$  the projection. We fix a  $G$ -equivariant connection on  $F$ , so that the operator  $\partial/\partial t$  acts on the sections of the restriction of  $F$  to  $U$ .

The main result of this appendix is the following

**Theorem 1.** *Assume that there exists a self-adjoint  $G$ -invariant transversally elliptic symmetric differential operator  $B : C^\infty(W, F) \rightarrow C^\infty(W, F)$ , whose restriction to  $U$  has the form*

$$B = \gamma \frac{\partial}{\partial t} + A, \quad (6)$$

where  $\gamma$  is a skew-adjoint bundle map, such that  $\gamma|_{E^\pm} = \pm\sqrt{-1}$ . Then the index  $\text{ind}^G A = 0$ .

*Remark 1.* By (5), the theorem is equivalent to the statement that  $\text{ind}_\rho^G A = 0$  for all  $\rho \in \hat{G}$ .

**2.4. The cobordism invariance.** Theorem 1 implies the cobordism invariance of the index in the following sense.

Assume that  $X_i, i = 0, 1$  are compact Riemannian  $G$ -manifolds and that  $E_i^\pm$  are  $G$ -equivariant Hermitian vector bundles over  $X_i$ . Let  $A_i^\pm : C^\infty(X_i, E_i^+) \rightarrow C^\infty(X_i, E_i^-)$  be transversally elliptic differential operators. Set  $A_i = A_i^+ \oplus A_i^-$ .

Suppose  $W$  is a compact  $G$ -manifold, whose boundary is the disjoint union of  $X_0$  and  $X_1$ . Then we can and we will identify a neighborhood  $U$  of the boundary of  $W$  with the disjoint union of the cylinders  $X_0 \times [0, \varepsilon)$  and  $X_1 \times (-\varepsilon, 0]$ . We denote by  $t : U \rightarrow \mathbb{R}$  the projection onto the second factor.

Assume that there exist a  $G$ -equivariant Hermitian vector bundle  $F$  over  $W$ , whose restriction to the boundary is isomorphic to the bundle induced by  $E_i = E_i^+ \oplus E_i^-$ , and an operator  $B : C^\infty(W, F) \rightarrow C^\infty(W, F)$ , which near the boundary takes the form (6). In this situation we say that the operators  $A_0$  and  $A_1$  are *cobordant*.

Let  $E_0^{\text{op}} = E_0^{\text{op}+} \oplus E_0^{\text{op}-}$  be the bundle  $E_0$  with the opposite grading, i.e.,  $E_0^{\text{op}\pm} = E_0^\mp$ . Then  $A_0$  defines the operator  $A_0^{\text{op}} : C^\infty(X_0, E_0^{\text{op}}) \rightarrow C^\infty(X_0, E_0^{\text{op}})$ . Clearly,  $\text{ind}^G A_0^{\text{op}} = -\text{ind}^G A_0$ .

Set  $X = X_0 \sqcup X_1$  and let  $E = E^+ \oplus E^-$  be the graded bundle over  $X$  induced by  $E_0^{\text{op}}$  and  $E_1$ . Let  $A : C^\infty(X, E) \rightarrow C^\infty(X, E)$  be the operator induced by  $A_0^{\text{op}}$  and  $A_1$ . Then the operators  $A$  and  $B$  satisfy the condition of Theorem 1. Hence,

$$\text{ind}^G A_1 - \text{ind}^G A_0 = \text{ind}^G A = 0.$$

Thus we proved the following

**Corollary 1.** *The distributional indexes of cobordant transversally elliptic operators coincide.*

Combining this corollary with the discussion in Subsection 1.2 we also obtain the following

**Corollary 2.** *The  $\text{Spin}^{\mathbb{C}}$ -quantization of orbifolds commutes with  $\text{Spin}^{\mathbb{C}}$ -cobordism.*

**2.5. The plan of the proof of Theorem 1.** We apply the method of [3] with necessary modifications.

Choose a  $G$ -invariant Riemannian metric on  $W$ , which induces the product metric on  $U = X \times (-\varepsilon, 0]$ . Let  $\tilde{W}$  denote the complete non-compact Riemannian manifold obtained from  $W$  by attaching the semi-infinite cylinder  $X \times [0, \infty)$  to the boundary. We extend the bundle  $F$  and the operator  $B$  to  $\tilde{W}$  in the obvious way.

Consider two linear operators  ${}^1 c_L$  and  $c_R$  on the exterior algebra  $\Lambda^\bullet \mathbb{C} = \Lambda^0 \mathbb{C} \oplus \Lambda^1 \mathbb{C}$ , defined by the formula

$$c_L \omega = 1 \wedge \omega - \iota_1 \omega; \quad c_R \omega = 1 \wedge \omega + \iota_1 \omega, \quad \omega \in \Lambda^\bullet \mathbb{C}. \quad (7)$$

(Here we consider 1 as a vector in  $\mathbb{C}$  and denote by  $\iota_1$  the interior multiplication by this vector). These operators anti-commute with each other,  $c_L c_R + c_R c_L = 0$ . They also satisfy  $c_L^2 = -1$ ,  $c_R^2 = 1$ .

Set  $\tilde{F} = F \otimes \Lambda^\bullet \mathbb{C}$  and consider the operator

$$\tilde{B} := \sqrt{-1} B \otimes c_L : C^\infty(\tilde{W}, \tilde{F}) \rightarrow C^\infty(\tilde{W}, \tilde{F}). \quad (8)$$

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<sup>1</sup>These operators generate two actions of the Clifford algebra of  $\mathbb{C}$  on  $\Lambda^\bullet \mathbb{C}$ , called, respectively, left and right actions. This is the motivation for the subscripts “ $L$ ” and “ $R$ ” in our notation.

Note, that the operator  $\tilde{B}$  is symmetric, since  $c_L^* = -c_L$ .

Let  $p : \tilde{W} \rightarrow \mathbb{R}$  be a  $G$ -invariant map, whose restriction to  $X \times (1, \infty)$  is the projection on the second factor, and such that  $p(W) = 0$  (see Subsection 4.1 for a convenient choice of this function). For any  $a \in \mathbb{R}$ , consider the operator  $\mathbf{B}_a := \tilde{B} - (p(x) - a) \otimes c_R$ . Then (cf. Lemma 1)

$$\mathbf{B}_a^2 = B^2 \otimes 1 - R + |p(x) - a|^2, \quad (9)$$

where  $R : C^\infty(\tilde{W}, \tilde{F}) \rightarrow C^\infty(\tilde{W}, \tilde{F})$  is a bounded operator.

Let  $\mathbf{B}_a^\pm$  denote the restriction of  $\mathbf{B}_a$  to the spaces  $F \otimes \Lambda^0 \mathbb{C}$  and  $F \otimes \Lambda^1 \mathbb{C}$  respectively. In Subsection 4.2 we show that, for each  $\rho \in \hat{G}$ , the index  $\text{ind}_\rho^G \mathbf{B}_a := \text{Ker}_\rho(\mathbf{B}_a^+) - \text{Ker}_\rho(\mathbf{B}_a^-)$  is well defined and is independent of  $a$ .

It follows from (9) that, if  $a \ll 0$ , then the operator  $\mathbf{B}_a^2$  is strictly positive. In particular, its kernel is empty and  $\text{ind}_\rho^G \mathbf{B}_a = 0$ . Also, if  $a \gg 0$ , then all the sections in  $\text{Ker} \mathbf{B}_a^2$  are concentrated on the cylinder  $X \times (0, \infty)$ , not far from  $X \times \{a\}$  (this part of the proof essentially repeats the arguments of Witten in [12]). Hence, the calculation of  $\text{Ker} \mathbf{B}_a^2$  is reduced to a problem on the cylinder  $X \times (0, \infty)$ . It is not difficult now to show that  $\text{ind}_\rho^G \mathbf{B}_a = \text{ind}_\rho^G A$  for  $a \gg 0, \rho \in \hat{G}$ .

Theorem 1 follows now from the fact that  $\text{ind}_\rho^G \mathbf{B}_a$  is independent of  $a$ .

### 3. TRANSVERSALLY ELLIPTIC OPERATORS AND THEIR INDEXES

In this section we recall the definitions and some properties of transversally elliptic operators on a compact  $G$ -manifold, cf. [1]. Since in the proof of Theorem 1 we apply some of the constructions which Atiyah used to prove that the index of a transversally elliptic operator is well defined, we will briefly recall this proof in Subsection 3.2.

Throughout the section  $X$  is a Riemannian  $G$ -manifold,  $E, F$  are  $G$ -equivariant Hermitian vector bundles over  $X$  and  $D : C^\infty(X, E) \rightarrow C^\infty(X, F)$  is a  $G$ -invariant pseudo-differential operator of order 1.

**3.1. Transversally elliptic operators.** Recall that the leading symbol  $\sigma(D)$  of  $D$  is a function on the cotangent bundle  $T^*X$  taking values in  $\text{Hom}(E, F)$ . Let  $T_G^*X \subset T^*X$  denote the subbundle of covectors which vanish on vectors tangent to the orbits of  $G$ . We will identify  $X$  with the zero section of  $T_G^*X$ .

**Definition 1.** *The operator  $D$  is called transversally elliptic if  $\sigma(D)$  is invertible when restricted to  $T_G^*X \setminus X$ .*

Fix a bi-invariant Riemannian metric on  $G$ , and let  $Y_1, \dots, Y_k$  be an orthonormal basis for the Lie algebra  $\mathfrak{g} = \text{Lie } G$ . Denote by  $\tilde{Y}_1, \dots, \tilde{Y}_k$  the corresponding first order differential operators defined by the action of  $G$  on  $E^+$ , and form the operator

$$\Delta_G := 1 - \sum_{i=1}^k \tilde{Y}_i^2 : C^\infty(X, E) \rightarrow C^\infty(X, E).$$

Consider the second order pseudo-differential operator

$$\bar{D} := (D, \Delta_G^{1/2}) : C^\infty(X, E) \rightarrow C^\infty(X, F) \oplus C^\infty(X, E).$$

One immediately sees that *the operator  $D$  is transversally elliptic if and only if the principal symbol of  $\bar{D}$  is injective on  $T^*X \setminus X$* . Equivalently, the operator

$$\bar{D}^* \bar{D} = D^* D + \Delta_G : C^\infty(X, E) \rightarrow C^\infty(X, E)$$

is elliptic (here  $D^*, \bar{D}^*$  denote the formal adjoints of the operators  $D$  and  $\bar{D}$ , respectively).

**3.2. The distributional index.** Suppose now that  $\rho \in \hat{G}$  is an irreducible representation of  $G$  and let

$$L_\rho^2(X, E) := \text{Hom}_G(\rho, L^2(X, E)) \otimes \rho$$

be the  $\rho$ -component of the space  $L^2(X, E^\pm)$  of square-integrable sections of  $E$ . Clearly, the restriction of  $\Delta_G$  to  $L_\rho^2(X, E)$  is bounded by a constant  $C(\rho)$ . It follows that the space  $\text{Ker}_\rho(D) := \text{Ker } D \cap L_\rho^2(X, E)$  is a subset of the space

$$(\bar{D}^* \bar{D})_{C(\rho)} := \{ u \in L_\rho^2(X, E) : \langle \bar{D}^* \bar{D} u, u \rangle \leq C(\rho) \}.$$

By the standard theory of elliptic operators (cf., for example, [9], [1, Lemma 2.3]), the space  $(\bar{D}^* \bar{D})_{C(\rho)}$  is finite dimensional. Hence, so is  $\text{Ker}_\rho(D)$  and we have the inequality

$$\dim \text{Ker}_\rho(D) \leq \dim (\bar{D}^* \bar{D})_{C(\rho)}.$$

Similarly,  $\dim \text{Ker}_\rho(D^*) \leq \dim (\bar{D} \bar{D}^*)_{C(\rho)}$ . With just a little more work (cf. [1, p. 13]), one shows that the sum (3) converges to a distribution on  $G$ . Thus the sum (4) also converges to a distribution on  $G$ , called the *distributional index* of  $A$ .

#### 4. INDEX OF THE OPERATOR $\mathbf{B}_a$

**4.1. The calculation of  $\mathbf{B}_a^2$ .** We will use the notation of Subsection 2.5. In particular,  $U \simeq X \times (-\varepsilon, 0]$  is a neighborhood of  $\partial W$ ,  $t : U \rightarrow (-\varepsilon, 0]$  is the projection,  $\tilde{W}$  is the manifold obtained from  $W$  by attaching a cylinder,  $\tilde{F} = F \otimes \Lambda^\bullet \mathbb{C}$  and  $\tilde{B}$  is the operator defined in (8). Recall that in Subsection 2.3 we have chosen a connection on  $F$ . This connection defines a trivialization of the restriction of  $F$  to  $U$  along the fibers of  $t$ . Hence the Hermitian metric on  $E$  induces a metric on  $F|_U$ . We extend this metric to a  $G$ -invariant Hermitian metric on  $F$ . This metric induces a Hermitian metric on  $\tilde{F}$  in the obvious way.

Let  $s : \mathbb{R} \rightarrow [0, \infty)$  be a smooth function such that  $s(t) = t$  for  $|t| \geq 1$ , and  $s(t) = 0$  for  $|t| \leq 1/2$ . Consider the map  $p : \tilde{W} \rightarrow \mathbb{R}$  such that  $p(y, t) = s(t)$  for  $(y, t) \in X \times (0, \infty)$  and  $p(x) = 0$  for  $x \in W$ . Recall that the operator  $c_R$  is defined in (7) and define the operator

$$\mathbf{B}_a := \tilde{B} - (p(x) - a) \otimes c_R. \quad (10)$$

**Lemma 1.** *Let  $\Pi_i : \tilde{F} \rightarrow F \otimes \Lambda^i \mathbb{C}$ , ( $i = 0, 1$ ) be the projections. Then*

$$\mathbf{B}_a^2 = B^2 \otimes 1 - R + |p(x) - a|^2, \quad (11)$$

where  $R : \tilde{F} \rightarrow \tilde{F}$  is a uniformly bounded bundle map, whose restriction to  $X \times (1, \infty)$  is equal to  $\sqrt{-1} \gamma(\Pi_1 - \Pi_0)$ , and whose restriction to  $W$  vanishes.

*Proof.* Note, first, that  $p(x) - a \equiv -a$  on  $W$ . Thus, since  $c_R$  anti-commutes with  $\tilde{B}$ , we have  $\mathbf{B}_a^2|_W = \tilde{B}^2|_W + a^2 = B^2 \otimes 1|_W + a^2$ . Hence, (11) holds, when restricted to  $W$ .

We now consider the restriction of  $\mathbf{B}_a^2$  to the cylinder  $X \times (0, \infty)$ . Since the operators  $c_L$  and  $c_R$  anti-commute, we obtain

$$\mathbf{B}_a^2|_{X \times (0, \infty)} = B^2 \otimes 1 + \sqrt{-1} s' \gamma \otimes c_L c_R + |s(t) - a|^2.$$

Since  $c_L c_R = \Pi_1 - \Pi_0$ , it follows, that (11) holds with  $R = s' \sqrt{-1} \gamma (\Pi_1 - \Pi_0)$ .  $\square$

**4.2. An estimate on the kernel of  $\mathbf{B}_a$ .** As in Subsection 3.1, we choose an orthonormal basis  $Y_1, \dots, Y_k$  for the Lie algebra  $\mathfrak{g} = \text{Lie } G$ , and we denote by  $\tilde{Y}_1, \dots, \tilde{Y}_k$  the corresponding first order differential operators defined by the action of  $G$  on  $C^\infty(\tilde{W}, \tilde{F})$ . Set  $\Delta_G := 1 - \sum_{i=1}^k \tilde{Y}_i^2$  and consider the second order pseudo-differential operator

$$\bar{\mathbf{B}}_a := (\mathbf{B}_a, \Delta_G^{1/2}) : C^\infty(\tilde{W}, \tilde{F}) \rightarrow C^\infty(\tilde{W}, \tilde{F}) \oplus C^\infty(\tilde{W}, \tilde{F}).$$

Using Lemma 1, we have

$$\bar{\mathbf{B}}_a^* \bar{\mathbf{B}}_a = \left( B^2 \otimes 1 + \Delta_G \right) + \left( |p(x) - a|^2 - R \right). \quad (12)$$

We consider  $\bar{\mathbf{B}}_a^* \bar{\mathbf{B}}_a$  as an operator acting on the space of square-integrable sections of  $\tilde{F}$ .

**Lemma 2.** *The operator  $\bar{\mathbf{B}}_a^* \bar{\mathbf{B}}_a$  is self-adjoint and has discrete spectrum.*

*Proof.* Since the operator  $B$  is transversally elliptic, the operator (12) is elliptic. Hence, cf., for example, [11, Lemma 6.3], the Lemma is equivalent to the following statement: For any  $\varepsilon > 0$  there exists a compact set  $K \subset \tilde{W}$ , such that if  $u$  is a smooth compactly supported section of  $\tilde{F}$ , then

$$\int_{\tilde{W} \setminus K} |u|^2 d\mu < \varepsilon \int_{\tilde{W}} \langle \bar{\mathbf{B}}_a^* \bar{\mathbf{B}}_a u, u \rangle d\mu. \quad (13)$$

Here,  $d\mu$  is the Riemannian volume element on  $\tilde{W}$ , and  $\langle \cdot, \cdot \rangle$  denotes the Hermitian scalar product on the fibers of  $\tilde{F}$ .

Set  $V(x) = |p(x) - a|^2 - R$ . To prove (13) note that, since  $R$  is bounded, there exists a compact set  $K \subset \tilde{W}$ , such that  $V > 1/\varepsilon$  on  $\tilde{W} \setminus K$ , i.e.,

$$\int_{\tilde{W} \setminus K} \langle Vu, u \rangle d\mu > \int_{\tilde{W} \setminus K} |u|^2 d\mu, \quad \text{for all } u \in L^2(\tilde{W}, \tilde{F}).$$

Note, also, that the first summand in (12) is a non-negative operator. Hence, we have

$$\int_{\tilde{W} \setminus K} |u|^2 d\mu < \varepsilon \int_{\tilde{W} \setminus K} \langle Vu, u \rangle d\mu \leq \varepsilon \int_{\tilde{W}} \langle Vu, u \rangle d\mu \leq \varepsilon \int_{\tilde{W}} \langle \bar{\mathbf{B}}_a^* \bar{\mathbf{B}}_a u, u \rangle d\mu. \quad \square$$

We apply the method of Subsection 3.2, to study the kernel of  $\mathbf{B}_a$ .

For an irreducible representation  $\rho \in \hat{G}$ , denote by  $L_\rho^2(\tilde{W}, \tilde{F})$  the  $\rho$ -component of the space of square-integrable sections, and by  $\text{Ker}_\rho(\mathbf{B}_a) = \text{Ker } \mathbf{B}_a \cap L_\rho^2(\tilde{W}, \tilde{F})$ .

**Lemma 3.** *The spectrum of the restriction of the operator  $\mathbf{B}_a$  to  $L_\rho^2(\tilde{W}, \tilde{F})$  is discrete. In particular,  $\dim \text{Ker}_\rho(\mathbf{B}_a) < \infty$ .*

*Proof.* As in Subsection 3.2, the equation (12) implies that, for each  $\rho \in \hat{G}$ , there is a constant  $C(\rho)$  such that, for all  $a, \lambda \in \mathbb{R}$  we have

$$\{u \in L_\rho^2(X, E) : \langle \mathbf{B}_a^2 u, u \rangle \leq \lambda\} \subset \{u \in L_\rho^2(X, E) : \langle \bar{\mathbf{B}}_a^* \bar{\mathbf{B}}_a u, u \rangle \leq C(\rho) + \lambda\}.$$

By Lemma 2, the dimension of the right hand side of this formula is finite. Hence, so is the dimension of the left hand side.  $\square$

Set  $\tilde{F}^+ := F \otimes \Lambda^0 \mathbb{C}$ ,  $\tilde{F}^- := F \otimes \Lambda^1 \mathbb{C}$ ,  $\mathbf{B}_a^\pm := \mathbf{B}_a|_{L^2(\tilde{W}, \tilde{F}^\pm)}$  and define

$$\text{ind}_\rho^G \mathbf{B}_a = \dim \text{Ker}_\rho(\mathbf{B}_a^+) - \dim \text{Ker}_\rho(\mathbf{B}_a^-). \quad (14)$$

*Remark 2.* One can obtain estimates on the growth of the numbers  $\text{ind}_\rho^G \mathbf{B}_a$  with  $\rho$  and prove that the distributional index  $\text{ind}^G \mathbf{B}_a$  is defined. The direct proof of this fact will be more involved than the proof in [1], since one have to work on a non-compact manifold  $\tilde{W}$ . We will show, however, that,  $\text{ind}_\rho^G \mathbf{B}_a = 0$  for all  $\rho \in \hat{G}$ ,  $a \in \mathbb{R}$ .

**Lemma 4.** *For each  $\rho \in \hat{G}$ , the index  $\text{ind}_\rho^G \mathbf{B}_a$  is independent of  $a$ .*

*Proof.* From (10), we see that  $\mathbf{B}_b - \mathbf{B}_a = (b - a) \otimes c_R$  is a bounded operator, depending continuously on  $b - a \in \mathbb{R}$ . Since  $\text{ind}_\rho^G \mathbf{B}_a$  coincides with the usual index of the restriction of  $\mathbf{B}^+$  to the space  $L_\rho^2(\tilde{W}, \tilde{F})$ , the lemma follows from the stability of the index of a Fredholm operator, cf., for example, [9, §I.8].  $\square$

**Lemma 5.**  $\text{ind}_\rho^G(\mathbf{B}_a) = 0$  for all  $\rho \in \hat{G}$ ,  $a \in \mathbb{R}$ .

*Proof.* By Lemma 4, it is enough to prove the proposition for one particular value of  $a$ . But it follows from Lemma 1 that, if  $a$  is a negative number such that  $a^2 > \sup_{x \in \tilde{W}} \|R(x)\|$ , then  $\mathbf{B}_a^2 > 0$ , so that  $\text{Ker} \mathbf{B}_a^2 = 0$ .  $\square$

To prove Theorem 1 it is enough now to show that  $\text{ind}_\rho^G \mathbf{B}_a = \text{ind}_\rho^G A$ . This is done in two steps: first, in Section 5, we construct a “model” operator  $\mathbf{B}^{\text{mod}}$  on the cylinder  $X \times (-\infty, \infty)$ , whose index is equal to  $\text{ind}_\rho^G A$ . Then, in Section 6, we show that  $\text{ind}_\rho^G \mathbf{B}_a = \text{ind}_\rho^G \mathbf{B}^{\text{mod}}$ .

## 5. THE MODEL OPERATOR

The bundles  $E^\pm$  lift to Hermitian vector bundles over the cylinder  $X \times \mathbb{R}$ , which we will denote by the same letters. Consider the Hermitian vector bundle  $\tilde{F} := (E^+ \oplus E^-) \otimes \Lambda^\bullet \mathbb{C}$  and the operator  $\mathbf{B}^{\text{mod}} : C^\infty(X \times \mathbb{R}, \tilde{F}) \rightarrow C^\infty(X \times \mathbb{R}, \tilde{F})$  defined by

$$\mathbf{B}^{\text{mod}} := \sqrt{-1} A \otimes c_L + \sqrt{-1} \gamma \otimes c_L \frac{\partial}{\partial t} + t \otimes c_R,$$

where  $t$  is the coordinate along the axis of the cylinder. We refer to  $\mathbf{B}^{\text{mod}}$  as the *model operator*, cf. [10]. It follows from Lemma 3 that the spectrum of the restriction of  $\mathbf{B}^{\text{mod}}$  to the space  $L_\rho^2(X \times \mathbb{R}, \tilde{F})$  is discrete (To see this, one can set  $W = X \times [0, 1]$ , and view  $X \times \mathbb{R}$  as a manifold obtained from  $W$  by attaching a cylinder).

We define  $\text{ind} \mathbf{B}^{\text{mod}}$  by (14).

**Lemma 6.** *The kernel of the model operator  $\mathbf{B}^{\text{mod}}$  is  $G$ -equivariantly isomorphic (as a graded space) to  $\text{Ker}(A)$ . In particular,  $\text{ind}_\rho^G \mathbf{B}^{\text{mod}} = \text{ind}_\rho^G A$  for all  $\rho \in \hat{G}$ .*

*Proof.* Repeating the arguments of Lemma 2 we see that the model operator  $\mathbf{B}^{\text{mod}}$  is self-adjoint. Hence, its kernel coincides with  $\text{Ker } \mathbf{B}^{\text{mod}^2}$ . Also from Subsection 3.2, we know that the kernel of the transversally elliptic operator  $A$  is a direct sum of the kernels of bounded operators  $A|_{L^2_\rho(X,E)}$ . It follows that  $\text{Ker } A = \text{Ker } A^2$ . Therefore, to prove the lemma it is enough to show that  $\text{Ker } \mathbf{B}^{\text{mod}^2}$  is equivariantly isomorphic to  $\text{Ker } A^2$ .

The same calculations as in the proof of Lemma 1 show that

$$(\mathbf{B}^{\text{mod}})^2|_{L^2(X \times \mathbb{R}, E^\pm \otimes \Lambda^\bullet \mathbb{C})} = A^2 \otimes 1 + 1 \otimes \left( -\frac{\partial^2}{\partial t^2} \pm (\Pi_1 - \Pi_0) + t^2 \right). \quad (15)$$

Both summands on the right hand side of (15) are non-negative. Hence, the kernel of  $(\mathbf{B}^{\text{mod}})^2$  is given by the tensor product of the kernels of these operators.

The space  $\text{Ker} \left( -\frac{\partial^2}{\partial t^2} + \Pi_1 - \Pi_0 + t^2 \right)$  is one dimensional and is spanned by the function  $\rho^+(t) := e^{-t^2/2} \in \Lambda^0 \mathbb{R}$ . Similarly,  $\text{Ker} \left( -\frac{\partial^2}{\partial t^2} + \Pi_0 - \Pi_1 + t^2 \right)$  is one dimensional and is spanned by the one-form  $\rho^-(t) := e^{-t^2/2} ds$ , where we denote by  $ds$  the generator of  $\Lambda^1 \mathbb{C}$ . It follows that

$$\text{Ker}(\mathbf{B}^{\text{mod}})^2 \cap L^2(X \times \mathbb{R}, E^\pm \otimes \Lambda^\bullet \mathbb{C}) \simeq \left\{ \sigma \otimes \rho^\pm(t) : \sigma \in \text{Ker } A^2 \cap L^2(X, E^\pm) \right\}.$$

□

**5.1. The shifted model operator.** Let  $T_a : X \times \mathbb{R} \rightarrow X \times \mathbb{R}$ ,  $T_a(x, t) = (x, t + a)$  be the translation, and consider the pull-back map  $T_a^* : L^2(X \times \mathbb{R}, \tilde{F}) \rightarrow L^2(X \times \mathbb{R}, \tilde{F})$ . Set

$$\mathbf{B}_a^{\text{mod}} := T_{-a}^* \circ \mathbf{B}^{\text{mod}} \circ T_a^* = B \otimes 1 - 1 \otimes c_R(t - a)$$

Then  $\text{ind}_\rho^G \mathbf{B}_a^{\text{mod}} = \text{ind}_\rho^G \mathbf{B}^{\text{mod}}$ , for any  $a, \rho \in \mathbb{R}$ .

## 6. PROOF OF THEOREM 1

In this section we fix  $\rho \in \hat{G}$ . All the operators studied in this section are restricted to the  $\rho$ -component of the space of square-integrable sections. For simplicity, we omit the subscript “ $\rho$ ” from the notion for these operators. In particular,  $\mathbf{B}_a^\pm$  denote the restriction of  $\mathbf{B}_a$  to the spaces  $L^2_\rho(\tilde{W}, \tilde{F}^\pm)$ . Similarly, let  $\mathbf{B}_{\pm,a}^{\text{mod}}, \mathbf{B}_{\pm,a}^{\text{mod}}$  denote the restriction of the operators  $\mathbf{B}^{\text{mod}}, \mathbf{B}_a^{\text{mod}}$  to the spaces  $L^2_\rho(X \times \mathbb{R}, \tilde{F}^\pm)$ . Note that, with this notation, we have

$$\text{ind}_\rho^G \mathbf{B}_a = \dim \text{Ker } \mathbf{B}_a^+ - \dim \text{Ker } \mathbf{B}_a^-; \quad \text{ind}_\rho^G \mathbf{B}_a^{\text{mod}} = \dim \text{Ker } \mathbf{B}_{+,a}^{\text{mod}} - \dim \text{Ker } \mathbf{B}_{-,a}^{\text{mod}}.$$

If  $A$  is a self-adjoint operator with discrete spectrum and  $\lambda \in \mathbb{R}$ , we denote by  $N(\lambda, A)$  the number of the eigenvalues of  $A$  not exceeding  $\lambda$  (counting multiplicities).

**Proposition 1.** *Let  $\lambda_\pm$  denote the smallest non-zero eigenvalue of  $(\mathbf{B}_\pm^{\text{mod}})^2$ . Then, for any  $0 < \varepsilon < \min\{\lambda_+, \lambda_-\}$ , there exists  $A = A(\varepsilon, V) > 0$ , such that*

$$N(\lambda_\pm - \varepsilon, (\mathbf{B}_a^\pm)^2) = \dim \text{Ker}(\mathbf{B}_{\pm,a}^{\text{mod}})^2, \quad \text{for all } a > A. \quad (16)$$

Before proving the proposition let us explain how it implies Theorem 1.

**6.1. Proof of Theorem 1.** Let  $V_{\varepsilon,a}^{\pm} \subset L_{\rho}^2(\tilde{W}, \tilde{F}^{\pm})$  denote the vector space spanned by the eigenvectors of the operator  $(\mathbf{B}_a^{\pm})^2$  with eigenvalues smaller or equal to  $\lambda_{\pm} - \varepsilon$ . The operator  $\mathbf{B}_a^{\pm}$  sends  $V_{\varepsilon,a}^{\pm}$  into  $V_{\varepsilon,a}^{\mp}$ . It follows that

$$\dim \operatorname{Ker} \mathbf{B}_a^+ - \dim \operatorname{Ker} \mathbf{B}_a^- = \dim V_{\varepsilon,a}^+ - \dim V_{\varepsilon,a}^-.$$

By Proposition 1, the right hand side of this equality equals  $\dim \operatorname{Ker} \mathbf{B}_+^{\operatorname{mod}} - \dim \operatorname{Ker} \mathbf{B}_-^{\operatorname{mod}}$ . Thus  $\operatorname{ind}_{\rho}^G \mathbf{B}_a = \operatorname{ind}_{\rho}^G \mathbf{B}^{\operatorname{mod}}$ . Theorem 1 follows now from Lemmas 5 and 6.  $\square$

The rest of this section is occupied with the proof of Proposition 1.

**6.2. Estimate from above on  $N(\lambda_{\pm} - \varepsilon, (\mathbf{B}_a^{\pm})^2)$ .** We will first show that

$$N(\lambda_{\pm} - \varepsilon, (\mathbf{B}_a^{\pm})^2) \leq \dim \operatorname{Ker} \mathbf{B}_{\pm}^{\operatorname{mod}}. \quad (17)$$

To this end we will estimate the operator  $\mathbf{B}_a^2$  from below. We will use the technique of [10, 4], adding some necessary modifications.

**6.3. The IMS localization.** Let  $j, \bar{j} : \mathbb{R} \rightarrow [0, 1]$  be smooth functions such that  $j^2 + \bar{j}^2 \equiv 0$  and  $j(t) = 1$  for  $t \geq 3$ , while  $j(t) = 0$  for  $t \leq 2$ . Set  $j_a(t) = j(a^{-1/2}t)$ ,  $\bar{j}_a(t) = \bar{j}(a^{-1/2}t)$ . This functions induce smooth functions on the cylinder  $X \times [0, 1]$ , which we denote by the same letters. By a slight abuse of notation we will denote by the same letters also the smooth functions on  $\tilde{W}$  given by the formulas  $j_a(x) = j(a^{-1/2}p(x))$ ,  $\bar{j}_a(x) = \bar{j}(a^{-1/2}p(x))$ .

The following version of IMS localization formula is due to Shubin [10, Lemma 3.1] (The abbreviation IMS stands for the initials of R. Ismagilov, J. Morgan, I. Sigal and B. Simon).

**Lemma 7.** *The following operator identity holds*

$$\mathbf{B}_a^2 = \bar{j}_a \mathbf{B}_a^2 \bar{j}_a + j_a \mathbf{B}_a^2 j_a + \frac{1}{2} [\bar{j}_a, [\bar{j}_a, \mathbf{B}_a^2]] + \frac{1}{2} [j_a, [j_a, \mathbf{B}_a^2]]. \quad (18)$$

*Proof.* Using the equality  $j_a^2 + \bar{j}_a^2 = 1$  we can write

$$\mathbf{B}_a^2 = j_a^2 \mathbf{B}_a^2 + \bar{j}_a^2 \mathbf{B}_a^2 = j_a \mathbf{B}_a^2 j_a + \bar{j}_a \mathbf{B}_a^2 \bar{j}_a + j_a [j_a, \mathbf{B}_a^2] + \bar{j}_a [\bar{j}_a, \mathbf{B}_a^2].$$

Similarly,  $\mathbf{B}_a^2 = \mathbf{B}_a^2 j_a^2 + \mathbf{B}_a^2 \bar{j}_a^2 = j_a \mathbf{B}_a^2 j_a + \bar{j}_a \mathbf{B}_a^2 \bar{j}_a - [j_a, \mathbf{B}_a^2] j_a - [\bar{j}_a, \mathbf{B}_a^2] \bar{j}_a$ . Summing these identities and dividing by 2, we come to (18).  $\square$

We will now estimate each of the summands in the right hand side of (18).

**Lemma 8.** *There exists  $A > 0$ , such that  $\bar{j}_a \mathbf{B}_a^2 \bar{j}_a \geq \frac{a^2}{8} \bar{j}_a^2$ , for all  $a > A$ .*

*Proof.* Note that  $p(x) \leq 3a^{1/2}$  for any  $x$  in the support of  $\bar{j}_a$ . Hence, if  $a > 36$ , we have  $\bar{j}_a^2 |p(x) - a|^2 \geq \frac{a^2}{4} \bar{j}_a^2$ .

Set  $A = \max \{ 36, 4 \sup_{x \in \tilde{W}} |R|^{1/2} \}$  and let  $a > A$ . Using Lemma 1, we obtain

$$\bar{j}_a \mathbf{B}_a^2 \bar{j}_a \geq \bar{j}_a^2 |p(x) - a|^2 - \bar{j}_a R \bar{j}_a \geq \frac{a^2}{8} \bar{j}_a^2. \quad \square$$

6.4. Let  $P_a : L^2_\rho(X \times \mathbb{R}, \tilde{F}) \rightarrow \text{Ker } \mathbf{B}_a^{\text{mod}}$  be the orthogonal projection. Let  $P_a^\pm$  denote the restriction of  $P_a$  to the space  $L^2_\rho(X \times \mathbb{R}, \tilde{F}^\pm)$ . Then  $P_a^\pm$  is a finite rank operator and its rank equals  $\dim \text{Ker } \mathbf{B}_{\pm, a}^{\text{mod}}$ . Clearly,

$$\mathbf{B}_{\pm, a}^{\text{mod}} + \lambda_\pm P_a^\pm \geq \lambda_\pm. \quad (19)$$

By identifying the support of  $j_a$  in  $X \times \mathbb{R}$  with a subset of  $\tilde{W}$ , we can and we will consider  $j_a P_a j_a$  and  $j_a \mathbf{B}_a^{\text{mod}} j_a$  as operators on  $\tilde{W}$ . Then  $j_a \mathbf{B}_a^2 j_a = j_a \mathbf{B}_a^{\text{mod}} j_a$ . Hence, (19) implies the following

**Lemma 9.**  $j_a \mathbf{B}_a^\pm j_a + \lambda_\pm j_a P_a^\pm j_a \geq \lambda_\pm j_a^2$ ,  $\text{rk } j_a P_a^\pm j_a \leq \dim \text{Ker } \mathbf{B}_\pm^{\text{mod}}$ .

For an operator  $A : L^2_\rho(\tilde{W}, \tilde{F}) \rightarrow L^2_\rho(\tilde{W}, \tilde{F})$ , we denote by  $\|A\|$  its norm.

**Lemma 10.** Let  $C = 2 \max \left\{ \max\{|j'(t)|^2, |\bar{j}'(t)|^2\} : t \in \mathbb{R} \right\}$ . Then

$$\| [j_a, [j_a, \mathbf{B}_a^2]] \| \leq C a^{-1}, \quad \| [\bar{j}_a, [\bar{j}_a, \mathbf{B}_a^2]] \| \leq C a^{-1}, \quad \text{for all } a > 0. \quad (20)$$

*Proof.* From Lemma 1 we obtain

$$\| [j_a, [j_a, \mathbf{B}_a^2]] \| = 2 |j'_a(t)|^2 = 2 a^{-1/2} |j'(a^{-1/2} t)|, \quad \| [\bar{j}_a, [\bar{j}_a, \mathbf{B}_a^2]] \| = 2 a^{-1/2} |\bar{j}'(a^{-1/2} t)|. \quad \square$$

From Lemmas 7, 9 and 10 we obtain the following

**Corollary 3.** For any  $\varepsilon > 0$ , there exists  $A = A(\varepsilon, V) > 0$ , such that, for all  $a > A$ , we have

$$\mathbf{B}_a^\pm + \lambda_\pm j_a P_a^\pm j_a \geq \lambda_\pm - \varepsilon, \quad \text{rk } j_a P_a^\pm j_a \leq \dim \text{Ker } \mathbf{B}_\pm^{\text{mod}}. \quad (21)$$

The estimate (17) follows from Corollary 3 and the following general lemma [8, p. 270]:

**Lemma 11.** Assume that  $A, B$  are self-adjoint operators in a Hilbert space  $\mathcal{H}$  such that  $\text{rk } B \leq k$  and there exists  $\mu > 0$  such that  $\langle (A + B)u, u \rangle \geq \mu \langle u, u \rangle$  for any  $u \in \text{Dom}(A)$ . Then  $N(\mu - \varepsilon, A) \leq k$  for any  $\varepsilon > 0$ .

6.5. **Estimate from below on  $N(\lambda_\pm - \varepsilon, (\mathbf{B}_a^\pm)^2)$ .** To prove Proposition 1 it remains now to show that

$$N(\lambda_\pm - \varepsilon, (\mathbf{B}_a^\pm)^2) \geq \dim \text{Ker } \mathbf{B}_{\pm, a}^{\text{mod}} \equiv \dim \text{Ker } \mathbf{B}_{\pm, a}^{\text{mod}}. \quad (22)$$

Let  $V_{\varepsilon, a}^\pm \subset L^2_\rho(\tilde{W}, \tilde{F})$  denote the vector space spanned by the eigenvectors of the operator  $(\mathbf{B}_a^\pm)^2$  with eigenvalues smaller or equal to  $\lambda_\pm - \varepsilon$ . Let  $\Pi_{\varepsilon, a}^\pm : L^2_\rho(\tilde{W}, \tilde{F}^\pm) \rightarrow V_{\varepsilon, a}^\pm$  be the orthogonal projection. Then  $\text{rk } \Pi_{\varepsilon, a}^\pm = N(\lambda_\pm - \varepsilon, (\mathbf{B}_a^\pm)^2)$ . As in Subsection 6.4, we can and we will consider  $j_a \Pi_{\varepsilon, a}^\pm j_a$  as an operator on  $L^2_\rho(X \times \mathbb{R}, \tilde{F}^\pm)$ . The proof of the following lemma does not differ from the proof of Corollary 3.

**Lemma 12.** For any  $\varepsilon > 0$ , there exists  $A = A(\varepsilon, V) > 0$ , such that, for any  $a > A$ , we have

$$\mathbf{B}_{\pm, a}^{\text{mod}} + \lambda_\pm j_a \Pi_{\varepsilon, a}^\pm j_a \geq \lambda_\pm - \varepsilon, \quad \text{rk } j_a \Pi_{\varepsilon, a}^\pm j_a \leq \dim N(\lambda_\pm - \varepsilon, (\mathbf{B}_a^\pm)^2). \quad (23)$$

The estimate (22) follows now from Lemmas 12 and 11.

The proof of Proposition 1 is complete.  $\square$

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