NOVIKOV TYPE INEQUALITIES FOR DIFFERENTIAL FORMS WITH NON-ISOLATED ZEROS

MAXIM BRAVERMAN AND MICHAEL FARBER

Abstract. We generalize the Novikov inequalities for 1-forms in two different directions: first, we allow non-isolated critical points (assuming that they are non-degenerate in the sense of R.Bott), and, secondly, we strengthen the inequalities by means of twisting by an arbitrary flat bundle. The proof uses Bismut's modification of the Witten deformation of the de Rham complex; it is based on an explicit estimate on the lower part of the spectrum of the corresponding Laplacian.

In particular, we obtain a new analytic proof of the degenerate Morse inequalities of Bott.

0. Introduction

0.1. **The Novikov inequalities.** Let M be a closed manifold of dimension n. In [N1, N2], S.P.Novikov associated to any real cohomology class $\xi \in H^1(M,\mathbb{R})$ a sequence of numbers $\beta_0(\xi),\dots,\beta_n(\xi)$ and proved that for any closed 1-form ω on M, having non-degenerate critical points, the following inequalities hold

$$m_p(\omega) \geq \beta_p(\xi), \qquad p=0,1,2,\dots \tag{0.1}$$

Here $\xi = [\omega] \in H^1(M,\mathbb{R})$ is the cohomology class of ω and $m_p(\omega)$ denotes the number of critical points of ω having Morse index p. Note also, that there are slightly stronger inequalities

$$\sum_{i=0}^{p}(-1)^i m_{p-i}(\omega) \ \geq\ \sum_{i=0}^{p}(-1)^i \beta_{p-i}(\xi), \qquad p=0,1,2,\dots, \tag{0.2}$$

cf. [F1]. In the case when the form ω is *exact* (i.e. $\omega = df$ where f is a non-degenerate Morse function) then $\xi = 0$ and the Novikov inequalities (0.2) turn into the classical Morse inequalities.

1991 *Mathematics Subject Classification.* Primary: 57R70; Secondary: 58A10.

Key words and phrases. Novikov inequalities, Morse-Bott inequalities, Witten deformation.

The research was supported by grant No. 449/94-1 from the Israel Academy of Sciences and Humanities.

A recent survey of the theory of Novikov inequalities may be found in [P2].

0.2. **Formulation of the main result.** In this paper we generalize the Novikov inequalities for 1-forms in two different directions: firstly, we allow non-isolated critical points and, secondly, we strengthen the inequalities by means of twisting by an arbitrary flat vector bundle.

Let M be a closed smooth manifold with a fixed flat complex vector bundle $\mathcal{F}$. Let ω be a smooth closed real valued 1-form on M, $d\omega = 0$, which is assumed to be *non-degenerate in the sense of R.Bott* [Bo1]. This means that the points of M, where the form ω vanishes form a submanifold of M (called the *critical points set* C of ω) and the *Hessian* of ω is *non-degenerate on the normal bundle to* C.

In order to make clear this definition, note that if we fix a tubular neighborhood N of C in M, then the monodromy of ω along any loop in N is obviously zero. Thus there exists a unique real valued smooth function f on N such that $df = \omega_{|N}$ and $f_{|C} = 0$. The *Hessian* of ω is then defined as the Hessian of f.

Let $\nu(C)$ denote the normal bundle of C in M. Note that $\nu(C)$ may have different dimension over different connected components of C. Since Hessian of ω is non-degenerate, the bundle $\nu(C)$ splits into the Whitney sum of two subbundles

$$\nu(C) \ = \ \nu^+(C) \ \oplus \nu^-(C), \tag{0.3}$$

such that the Hessian is strictly positive on $\nu^+(C)$ and strictly negative on $\nu^-(C)$. Here again, the dimension of the bundles $\nu^+(C)$ and $\nu^-(C)$ over different connected components of the critical point set may be different.

For every connected component Z of the critical point set C, the dimension of the bundle $\nu^-(C)$ over Z is called the *index* of Z (as a critical submanifold of ω) and is denoted by $\operatorname{ind}(Z)$. Let $o(Z)$ denote the *orientation bundle of* $\nu^-(C)_{|Z}$*, considered as a flat line bundle.* Consider the *twisted Poincaré polynomial* of component Z

$$\mathcal{P}_{Z,\mathcal{F}}(\lambda) \ = \ \sum_{i=0}^{\dim Z} \lambda^i \dim_{\mathbb{C}} H^i(Z, \mathcal{F}_{|Z} \otimes o(Z)) \tag{0.4}$$

(here $H^i(Z, \mathcal{F}_{|Z} \otimes o(Z))$ denote the cohomology of Z with coefficients in the flat vector bundle $\mathcal{F}_{|Z} \otimes o(Z))$ and define using it the following *Morse counting polynomial*

$$\mathcal{M}_{\omega,\mathcal{F}}(\lambda) \ = \ \sum_{Z} \lambda^{\operatorname{ind}(Z)} \mathcal{P}_{Z,\mathcal{F}}(\lambda), \tag{0.5}$$

where the sum is taken over all connected components Z of C.

On the other hand, with one-dimensional cohomology class $\xi = [\omega] \in H^1(M, \mathbb{R})$ and the flat vector bundle $\mathcal{F}$, one can associate canonically the *Novikov counting polynomial*

$$\mathcal{N}_{\xi,\mathcal{F}}(\lambda) \;=\; \sum_{i=0}^{n} \lambda^i \beta_i(\xi, \mathcal{F}), \tag{0.6}$$

where $\beta_i(\xi, \mathcal{F})$ are generalizations of the Novikov numbers, cf. Definition 1.2. Note, that if $\xi = 0$ and $\mathcal{F}$ is the trivial line bundle, then $\mathcal{N}_{\xi,\mathcal{F}}(\lambda)$ coincides with the Poincaré polynomial of M.

The following is our principal result.

Theorem 0.3. *In the situation described above, there exists a polynomial*

$$\mathcal{Q}(\lambda) = q_0 + q_1\lambda + q_2\lambda^2 + \dots$$

with non-negative integer coefficients $q_i \geq 0$, such that

$$\mathcal{M}_{\omega,\mathcal{F}}(\lambda) \;-\; \mathcal{N}_{\xi,\mathcal{F}}(\lambda) \;=\; (1+\lambda)\mathcal{Q}(\lambda). \tag{0.7}$$

The main novelty in this theorem is that it is applicable to the case of 1-forms with non-isolated singular points. Thus, we obtain, in particular, a new proof of the degenerate Morse inequalities of R.Bott. Moreover, Theorem 0.3 provides a generalization of the Morse-Bott inequalities to the case of an arbitrary flat vector bundle $\mathcal{F}$; this generally produces stronger inequalities as shown in Section 1.7.

Next, we are going to point out the following corollary.

Corollary 0.4 (Euler-Poincaré theorem)**.** *Under the conditions of Theorem 0.3, the Euler characteristic of M can be computed as*

$$\chi(M) \;=\; \sum_{Z} (-1)^{\operatorname{ind}(Z)} \chi(Z), \tag{0.8}$$

where the sum is taken over all connected components $Z \subset C$.

The corollary is obtained from (0.7) by substituting $\lambda = -1$ and observing that

$$\mathcal{M}_{\omega,\mathcal{F}}(-1) = d \cdot \sum_{Z} (-1)^{\operatorname{ind}(Z)} \chi(Z), \qquad \mathcal{N}_{\xi,\mathcal{F}}(-1) = d \cdot \chi(M), \tag{0.9}$$

where $d = \dim \mathcal{F}$; the last equality in (0.9) follows immediately from Definition 1.2.

0.5. **The case of isolated critical points.** Let's consider the special case when all critical points of ω are isolated. Then the Morse counting polynomial (0.5) takes the form

$$\mathcal{M}_{\omega,\mathcal{F}}(\lambda) \ = \ d\cdot\sum_{p=0}^{n}\lambda^p m_p(\omega), \tag{0.10}$$

where $d=\dim\mathcal{F}$ and $m_p(\omega)$ denotes the number of critical points of ω of index p. Theorem 0.3 gives in this case the inequalities

$$\sum_{i=0}^{p}(-1)^i m_{p-i}(\omega) \ \geq \ d^{-1}\cdot\sum_{i=0}^{p}(-1)^i\beta_{p-i}(\xi,\mathcal{F}), \qquad p=0,1,2,\dots \tag{0.11}$$

The last inequalities coincide with the Novikov inequalities (0.2) in the special case when $\mathcal{F}=\mathbb{R}$ with the trivial flat structure. Easy examples described in Section 1.7, show that using of the flat vector bundle $\mathcal{F}$ gives sharper estimates in general, than the standard approach with $\mathcal{F}=\mathbb{R}$.

On the other hand, (0.11) clearly generalizes the Morse type inequalities obtained by S.P.Novikov in [N3], using Bloch homology (which correspond to the case, when $[\omega]=0\in H^1(M,\mathbb{R})$ in (0.11)).

0.6. **The method of the proof.** Our proof of Theorem 0.3 is based on a slight modification of the Witten deformation [W] suggested by Bismut [B] in his proof of the degenerate Morse inequalities of Bott. However our proof is rather different from [B] even in the case $[\omega]=0$. We entirely avoid the probabilistic analysis of the heat kernels, which is the most difficult part of [B]. Instead, we give an explicit estimate on the number of the "small" eigenvalues of the deformed Laplacian. We now will explain briefly the main steps of the proof.

Let $\Omega^\bullet(M,\mathcal{F})$ denote the space of smooth differential forms on M with values in $\mathcal{F}$. In Section 4, we introduce a 2-parameter deformation

$$\nabla_{t,\alpha}:\Omega^\bullet(M,\mathcal{F})\to\Omega^{\bullet+1}(M,\mathcal{F}), \qquad t,\alpha\in\mathbb{R} \tag{0.12}$$

of the covariant derivative ∇, such that, for large values of t,α the Betti numbers of the deformed de Rham complex $\big(\Omega^\bullet(M,\mathcal{F}),\nabla_{t,\alpha}\big)$ are equal to the Novikov numbers $\beta_p(\xi,\mathcal{F})$. Outside of a small tubular neighborhood of the critical points set C of ω the differential (0.12) is given by the formula

$$\nabla_{t,\alpha}:\theta\mapsto\theta+t\alpha\omega\wedge\theta, \qquad \theta\in\Omega^\bullet(M,\mathcal{F}).$$

Next we construct a special Riemannian metric g^M on M. In fact, we, first, chose a Riemannian metric on the normal bundle $\nu(C)$ and, then, extend it to a metric on M. We also choose a Hermitian metric $h^{\mathcal{F}}$ on

$\mathcal{F}$. Let us denote by $\Delta_{t,\alpha}$ the Laplacian associated with the differential (0.12) and with the metrics $g^M, h^{\mathcal{F}}$.

Fix $\alpha > 0$ sufficiently large. It turns out that, when $t \to \infty$, the eigenfunctions of $\Delta_{t,\alpha}$ corresponding to "small" eigenvalues localize near the critical points set C of ω. Hence, the number of the "small" eigenvalues of $\Delta_{t,\alpha}$ may be calculated by means of the restriction of $\Delta_{t,\alpha}$ on a tubular neighborhood of C. This neighborhood may be identified with a neighborhood of the zero section of the normal bundle $\nu(C)$ to C. We are led, thus, to study of a certain Laplacian on $\nu(C)$. The latter Laplacian may be decomposed as $\bigoplus_Z \Delta^Z_{t,\alpha}$ where the sum ranges over all connected components of C and $\Delta^Z_{t,\alpha}$ is a Laplacian on the normal bundle $\nu(Z) = \nu(C)_{|Z}$ to Z. We denote by $\Delta^{Z,p}_{t,\alpha}$ $(p = 0, 1, 2, \dots)$ the restriction of $\Delta^Z_{t,\alpha}$ on the space of p-forms.

The operator (0.12) is constructed so that the spectrum of $\Delta^Z_{t,\alpha}$ does not depend on t. Moreover, if $\alpha > 0$ is sufficiently large, then

$$\dim \operatorname{Ker} \Delta^{Z,p}_{t,\alpha} = \dim H^{p-\operatorname{ind}(Z)}(Z, \mathcal{F}_{|Z}). \tag{0.13}$$

In the case when $\mathcal{F}$ is a trivial line bundle, the equation (0.13) is proven by Bismut [B, Theorem 2.13]. We prove (0.13) in Section 3. Note that our proof is rather different from [B].

Let $E^\bullet_{t,\alpha}$ $(p = 0, 1, \dots, n)$ be the subspace of $\Omega^\bullet(M, \mathcal{F})$ spanned by the eigenvectors of $\Delta_{t,\alpha}$ corresponding to the "small" eigenvalues. The cohomology of the deformed de Rham complex $\big(\Omega^\bullet(M, \mathcal{F}), \nabla_{t,\alpha}\big)$ may be calculated as the cohomology of the subcomplex $\big(E^\bullet_{t,\alpha}, \nabla_{t,\alpha}\big)$.

We prove (Theorem 4.8) that, if the parameters t and α are large enough, then

$$\dim E^p_{t,\alpha} = \sum_Z \dim \operatorname{Ker} \Delta^{Z,p}_{t,\alpha}, \tag{0.14}$$

where the sum ranges over all connected components Z of C. The Theorem 0.3 follows now from (0.13),(0.14) by standard arguments (cf. [Bo2]).

Remark 0.7. In [HS3], Helffer and Sjöstrand gave a very elegant analytic proof of the degenerate Morse inequalities of Bott. Though they also used the ideas of [W], their method is completely different from [B]. It is not clear if this method may be applied to the case $\xi \neq 0$.

0.8. **Contents.** The paper is organized as follows.

In Section 1, we define a slightly generalized version of the Novikov numbers associated to a cohomology class $\xi \in H^1(M, \mathbb{R})$ and a flat vector bundle $\mathcal{F}$. Here we also discuss some examples.

In Section 2, we recall the construction of Bismut's deformation of the de Rham complex on a fiber bundle and discuss the spectral properties of the corresponding Laplacian. At the end of the section we state Theorem 2.12 which calculates the kernel of this Laplacian.

In Section 3, we prove Theorem 2.12.

In Section 4, we define the Bismut deformation of the Laplacian on M, taking into account that ω is not cohomologious to 0. Then we prove that the number of the eigenvalues of this Laplacian which tend to 0 as the parameter tends to infinity is equal to the dimension of the kernel of some Laplacian on $\nu(Z)$.

In Section 5, we prove the main Theorem 0.3.

The results contained in this paper were announced in [BF].

1. The Novikov numbers

In this section we recall the definition and the main properties of the Novikov numbers [N1, N2] associated to a cohomology class $\xi \in H^1(M,\mathbb{R})$. In fact, we define these numbers in a slightly more general situation. Our point of view is motivated by the study of deformations of elliptic complexes in [F2]. Roughly speaking, any one-dimensional cohomology class defines an analytic deformation of the twisted de Rham complex and the Novikov numbers are the instances of the natural invariants of such deformations, cf. [F2].

1.1. **The Novikov deformation.** Let M be a closed smooth manifold and let $\mathcal{F}$ be a complex flat vector bundle over M. We will denote by

$$\nabla : \Omega^\bullet(M,\mathcal{F}) \to \Omega^{\bullet+1}(M,\mathcal{F})$$

the covariant derivative on $\mathcal{F}$.

Given a closed 1-form $\omega \in \Omega^1(M)$ on M with real values, it determines a family of connections on $\mathcal{F}$ (the *Novikov deformation*) parameterized by the real numbers $t \in \mathbb{R}$

$$\nabla_t : \Omega^i(M,\mathcal{F}) \to \Omega^{i+1}(M,\mathcal{F}), \tag{1.1}$$

where

$$\nabla_t\theta = \nabla\theta + t\omega\wedge\theta, \qquad \theta \in \Omega^\bullet(M,\mathcal{F}). \tag{1.2}$$

All the connections ∇_t are flat, i.e. $\nabla_t^2 = 0$, if the form ω is closed, $d\omega = 0$.

We can view the obtained complex as follows. For $t \in \mathbb{R}$, let ρ_t denote the flat real line bundle over M with the monodromy representation

$\rho_t : \pi_1(M) \to \mathbb{R}^*$ given by the formula

$$\rho_t(\gamma) = \exp(-t\int_\gamma \omega) \quad \in \ \mathbb{R}^*, \qquad \gamma \in \pi_{\nvDash}(\mathbb{M}). \tag{1.3}$$

Then ∇_t can be considered as the covariant derivative on the flat bundle $\mathcal{F} \otimes \rho_t$.

Note that changing ω by a cohomologious 1-form determines a gauge equivalent connection ∇_t and so the cohomology $H^\bullet(M, \mathcal{F} \otimes \rho_t)$ depends only on the cohomology class $\xi = [\omega] \in H^1(M, \mathbb{R})$ of ω.

The dimension of the cohomology $H^i(M, \mathcal{F} \otimes \rho_t)$ is an integer valued function of $t \in \mathbb{R}$ having the following behavior. There exists a discrete subset $S \subset \mathbb{R}$ (i.e. each of its points is isolated) such that the dimension $\dim\ H^i(M, \mathcal{F} \otimes \rho_t)$ is *constant for* $t \notin S$ (the corresponding value of the dimension we will call the *background value*) and for $t \in S$ the dimension of $H^i(M, \mathcal{F} \otimes \rho_t)$ is *greater* than the background value. Cf., for example, [F2, Theorem 2.8], where a more precise information for the case of elliptic complexes is given. The subset S above will be called *the set of jump points.*

Definition 1.2. For each $i = 0, 1, \dots, n$, the background value of the dimension of $H^i(M, \mathcal{F} \otimes \rho_t)$ is called the i-th *Novikov number* $\beta_i(\xi, \mathcal{F})$.

The novelty here is in introduction of the flat vector bundle $\mathcal{F}$; the standard definition uses the trivial line bundle (over $\mathbb{C}$) instead of our $\mathcal{F}$. The importance of this generalization will be explained in Section 1.7 below.

Lemma 1.3. *The set of jump points S is finite.*

Proof. Given class $\xi \in H^1(M, \mathbb{R})$, consider the set $\mathcal{R}(\xi)$ of all cohomology classes $\rho \in H^1(M, \mathbb{R})$ such that the corresponding period map $\rho_* : H_1(M, \mathbb{Z}) \to \mathbb{R}$ vanishes on the kernel of the period map $\xi_* : H_1(M, \mathbb{Z}) \to \mathbb{R}$. The image of the period map $\xi_* : H_1(M, \mathbb{Z}) \to \mathbb{R}$ is isomorphic to $\mathbb{Z}^{\lessdot}$ for some l (which is called the *degree of irrationality of* ξ). Any element $\rho \in \mathcal{R}(\xi)$ determines a representation of the fundamental group

$$\pi = \pi_1(M) \ \to \mathbb{R}^{>\nvDash}, \qquad \text{where} \quad \eth \mapsto \exp(-\rho_*[\eth]) \ \in \mathbb{R}^{>\nvDash}.$$

This allows to identify $\mathcal{R}(\xi)$ with $(\mathbb{R}^*)^{\lessdot}$ and thus we introduce the structure of affine algebraic variety in $\mathcal{R}(\xi)$.

Consider the function $\dim_{\mathbb{C}} H^i(M, \mathcal{F} \otimes \rho)$ as a function of $\rho \in \mathcal{R}(\xi)$. The standard arguments show that there exist an algebraic $V \subset neq(\mathbb{R}^*)^{\lessdot}$ such that $\dim_{\mathbb{C}} H^i(M, \mathcal{F} \otimes \rho)$ is constant for all $\rho \notin V$ (cf. [Ha, Ch.3 §12]).

If we identify $\mathcal{R}(\xi)$ with $(\mathbb{R}^*)^{\lessdot}$ as explained above, then the point ξ will be represented by a vector $(e^{a_1}, e^{a_2}, \dots, e^{a_l})$ with the real numbers $a_1, a_2, \dots, a_l$ linearly independent over $\mathbb{Q}$. Then, for $t \in \mathbb{R}$, the class $t\xi$ (describing the 1-dimensional local system ρ_t) is represented by the vector $(e^{ta_1}, e^{ta_2}, \dots, e^{ta_l}) \in (\mathbb{R}^*)^{\lessdot}$. It is easy to see that the curve

$$\psi(t) \;=\; (e^{ta_1}, e^{ta_2}, \dots, e^{ta_l}) \in (\mathbb{R}^*)^{\lessdot}$$

has the following property: for any algebraic subvariety $W \subset (\mathbb{R}^*)^{\lessdot}$ there exists a constant $C > 0$ such that $\psi(t) \notin W$ for $|t| > C$.

This show that $\dim_{\mathbb{C}} H^i(M, \mathcal{F} \otimes \rho_t)$ is equal to the background value for t sufficiently large. Thus the set of jump points S is finite. □

1.4. **The monodromy of the deformed connection ∇_t.** Let $\mathcal{F}_0$ denote the fiber of $\mathcal{F}$ over the base point $*$ of M. Let $\phi : \pi_1(M) \to \mathrm{GL}_{\mathbb{C}}(\mathcal{F}_0)$ denote the monodromy representation of the flat bundle $\mathcal{F}$. Then the monodromy representation of the flat vector bundle $\mathcal{F} \otimes \rho_t$ is given by

$$g \mapsto \exp(-t \int_g \omega)\phi(g) \;\in \mathrm{GL}_{\mathbb{C}}(\mathcal{F}_0) \qquad \text{for} \qquad g \in \pi_1(M).$$

This formula describes explicitly the *deformation of the monodromy representation*.

1.5. **Computation of the Novikov numbers in terms of the spectral sequence.** One may compute the Novikov numbers $\beta_i(\xi, \mathcal{F})$ by means of the cell structure of M as the dimension of the homology of a local system over M determined by the deformation (1.2); the dimension here is understood over the field of germs of meromorphic curves in $\mathbb{C}$. This was explained in [F2] and in [FL] in terms of the *germ complex of the deformation*; we will not repeat this construction here. It leads naturally to the computation of the Novikov numbers by means of a spectral sequence, cf. Theorems 2.8 and 6.1 in [F2]. It is important to point out that both these approaches (the one, based on the germ complex, and the spectral sequence) are able to find the Novikov numbers *starting from an arbitrary value $t = t_0$ and using infinitesimal information on the deformation*.

Note, that a spectral sequence of a similar nature appears in [P2].

We will briefly describe the spectral sequence. Fix an arbitrary value $t = t_0 \in \mathbb{R}$. Then there exists a spectral sequence $E_r^*, \;\; r \geq 1$ with the following properties:

(1) The initial term of the spectral sequence coincides with the cohomology at the chosen point $t = t_0$:

$$E_1^i \;= H^i(M, \mathcal{F} \otimes \rho_{t_0});$$

(2) For large r all differentials of the spectral sequence $d_r : E_r^i \to E_r^{i+1}$ vanish and the limit term E_∞^i is isomorphic to the background cohomology;

(3) The first differential
$$H^i(M, \mathcal{F} \otimes \rho_{t_0}) \ \to H^{i+1}(M, \mathcal{F} \otimes \rho_{t_0})$$
is given by multiplication by $\xi \in H^1(M, \mathbb{R})$;

(4) The higher differentials are given by the iterated Massey products with ξ.

The spectral sequence is constructed as follows, cf.[F2, Section 6.1]. Denote by Z_r^i the set of polynomials of the form
$$f(t) = f_0 + tf_1 + t^2 f_2 + \cdots + t^{r-1} f_{r-1}$$
where $f_j \in \Omega^i(M, \mathcal{F} \otimes \rho_{t_0})$ such that $\nabla_t f(t)$ is divisible by $(t - t_0)^r$. Then
$$E_r^i \ = \ Z_r^i / (tZ_{r-1}^i + t^{1-r} \nabla_t Z_{r-1}^{i-1})$$
and the differential
$$d_r : E_r^i \ \to \ E_r^{i+1}$$
is induced by the action of $t^{-r}\nabla_t$. See [F2, Section 6], for more detail.

Corollary 1.6. *If $\xi \in H^1(M, \mathbb{R})$ is a non-zero class then the zero-dimensional Novikov number $\beta_0(\xi, \mathcal{F})$ vanishes.*

It follows directly by applying the spectral sequence.

1.7. **Some examples.** Here we will produce examples, where the Novikov numbers twisted by a flat vector bundle $\mathcal{F}$ (as defined above) give greater values (and thus stronger inequalities) than the usual Novikov numbers (where $\mathcal{F} = \mathbb{R}$ or $\mathbb{C}$, cf. [N1, N2, P2]).

Let $k \subset S^3$ be a smooth knot and let the 3-manifold X be the result of 1/0-surgery on S^3 along k. Note that the one-dimensional homology group of X is infinite cyclic and thus for any complex number $\eta \in \mathbb{C}$, $\eta \neq 0$, there is a complex flat line bundle over X such that the monodromy with respect to the generator of $H_1(X)$ is η. Denote such flat bundle by $\mathcal{F}_\eta$.

Note also, that for $\eta \neq 1$, the dimension of $H^1(X, \mathcal{F}_\eta)$ is equal to the multiplicity of η as a root of the Alexander polynomial of the knot k, cf. [R]. Thus, by a choice of the knot k and the number $\eta \in \mathbb{C}^*$, we may make the group $H^1(X, \mathcal{F}_\eta)$ arbitrarily large, while $H^1(X, \mathbb{C})$ is always one-dimensional. (One may, for example, take multiple connected sum of many copies of a knot with non-trivial Alexander polynomial).

Consider now the 3-manifold M which is the connected sum
$$M \ = \ X \ \# \ (S^1 \times S^2).$$

Thus $M = X_+ \cup X_-$ where $X_+ \cap X_- = X_0 = S^2$ and

$$X_+ = X - \{disk\} \quad \text{and} \quad X_- = (S^1 \times S^2) - \{disk\}.$$

Consider a flat complex line bundle $\mathcal{F}$ over M such that its restriction over X_+ is isomorphic to $\mathcal{F}_\eta|_{X_+}$. Consider the class $\xi \in H^1(X, \mathbb{R})$ such that its restriction onto X_+ is trivial and its restriction to X_- is the generator.

We want to compute the Novikov number $\beta_1(\xi, \mathcal{F})$. By using the Mayer-Vietoris sequence, we obtain, for generic t,

$$H^1(M, \mathcal{F} \otimes \rho_t) \;\simeq\; H^1(X_+, \mathcal{F}_\eta) \oplus H^1(X_-, \mathcal{F} \otimes \rho_t)$$

For generic t, the second term vanishes and thus we obtain

$$\beta_1(\xi, \mathcal{F}) \;=\; \dim_{\mathbb{C}} H^1(X, \mathcal{F}_\eta)$$

As we noticed above, this number can be arbitrarily large, while $\dim_{\mathbb{C}} H^1(M, \mathbb{C}) = 2$.

2. The Bismut Laplacian on a fiber bundle

In this section we describe a version of the Bismut deformation of the Laplacian on the bundle E, considered as a non-compact manifold. In Section 4 we will apply this construction to the case when E is the normal bundle to a connected submanifold of the set of critical points of ω.

2.1. **Description of the data.** Let $E = E^+ \oplus E^-$ be a $\mathbb{Z}_{\not{2}}$-graded finite dimensional vector bundle over a compact connected manifold Z.

Suppose that $\mathcal{F}$ is a flat vector bundle over E. We denote by $\Omega^\bullet(E, \mathcal{F})$ the space of differential forms on E with values in $\mathcal{F}$ and by $\nabla : \Omega^\bullet(E, \mathcal{F}) \to \Omega^{\bullet+1}(E, \mathcal{F})$ the corresponding covariant derivative operator.

2.2. **A splitting of the tangent space.** Fix an Euclidean metric h^E on the bundle E (i.e. a smooth fiberwise metric) such that E^+ and E^- are orthogonal. For a vector $y \in E$, we will denote by $|y|$ its norm with respect to the metric h^E.

Next choose an Euclidean connection ∇^E on E which preserves the decomposition $E = E^+ \oplus E^-$. Then the tangent space TE splits naturally into

$$TE = T^H E \oplus T^V E, \tag{2.1}$$

where T^VE is the set of the vectors in TE which are tangent to the fibers of E (the *vertical vectors*), and T^HE is the set of *horizontal vectors* in TE.

2.3. **A Riemannian metric on E.** In this section we consider E as a non-compact manifold. Our aim is to introduce a Riemannian metric on E. Let π be the projection $E \to Z$. If $y \in E$, then π_* identifies T_y^HE with $T_{\pi(y)}Z$. Choose any Riemannian metric g^Z on Z. Then T_y^HE is naturally endowed with the metric π^*g^Z. Also T^VE and E can be naturally identified. Hence, the metric h^E on E induces a metric on T^VE. We still denote this metric by h^E and we define the Riemannian metric

$$g^E = h^E \oplus \pi^* g^Z \tag{2.2}$$

on TE, which coincides with h^E on T^VE, with π^*g^Z on T^HV and such that T^HE and T^VE are orthogonal. Note that the metric g^E depends upon the choices of h^E, g^Z and ∇^E.

2.4. **An Euclidean metric on $\mathcal{F}$.** We will identify the manifold Z with the zero section of E. Let $\mathcal{F}_{|Z}$ denote the restriction of $\mathcal{F}$ on Z. Fix an arbitrary Euclidean metric h on $\mathcal{F}_{|Z}$. The flat connection on $\mathcal{F}$ defines a trivialization of $\mathcal{F}$ along the fibers of E and, hence, gives a natural extension of h to an Euclidean metric $h^{\mathcal{F}}$ on $\mathcal{F}$ which is flat along the fibers of E.

2.5. **A bigrading on the space of differential forms.** The metrics $g^E, h^{\mathcal{F}}$ define an L_2-scalar product on the space of differential forms on E with values in $\mathcal{F}$. Let $\Omega_{(2)}(E,\mathcal{F}) = \bigoplus \Omega^p_{(2)}(E,\mathcal{F})$ denote the Hilbert space of square integrable differential forms on E with values in $\mathcal{F}$.

The splitting (2.1) and the corresponding decomposition $T^*E = (T^HE)^* \oplus (T^VE)^*$ induce a bigrading on $\Omega_{(2)}(E,\mathcal{F})$ by

$$\Omega_{(2)}(E,\mathcal{F}) = \bigoplus_{i,j} \Omega^{i,j}_{(2)}(E,\mathcal{F}), \tag{2.3}$$

where $\Omega^{i,j}_{(2)}(E,\mathcal{F})$ is the space of square integrable sections of

$$\Lambda^i\big((T^HE)^*\big) \otimes \Lambda^j\big((T^VE)^*\big) \otimes \mathcal{F}.$$

Here, as usual, we denote by $\Lambda(V) = \bigoplus \Lambda^p(V)$ the exterior algebra of a vector space V.

For $s \neq 0$, let τ_s be the map from $\Omega_{(2)}(E,\mathcal{F})$ onto itself, which sends $\alpha \in \Omega^{i,j}_{(2)}(E,\mathcal{F})$ to $s^j\alpha$. Of course, τ_s depends upon the connection on E given by (2.1).

2.6. **The Bismut complex on a fiber bundle.** In this section we produce a family of complexes depending on $t > 0$, which is a special case of the construction of Bismut [B, Section 2(b)]. Note that our notations are slightly different from [B].

Let $f : E \to \mathbb{R}$ denote the function such that its value on a vector $y = (y^+, y^-) \in E^+ \oplus E^-$ is given by

$$f(y) = \frac{|y^+|^2}{2} - \frac{|y^-|^2}{2}. \tag{2.4}$$

Recall that $\nabla : \Omega^\bullet_{(2)}(E, \mathcal{F}) \to \Omega^{\bullet+1}_{(2)}(E, \mathcal{F})$ denotes the covariant differential operator determined by the flat connection on $\mathcal{F}$. Following Bismut [B], we define a family of differentials

$$\nabla_{t,\alpha} = (\tau_{\sqrt{t}})^{-1} e^{-\alpha t f} \nabla e^{\alpha t f} \tau_{\sqrt{t}}, \qquad t > 0,\ \alpha > 0. \tag{2.5}$$

2.7. **An alternative description of $\nabla_{t,\alpha}$.** We will also need another description of the differential (2.5) (cf. [B, Remark 2]).

For $s > 0$, let $r_s : E \to E$ be the multiplication by s, i.e $r_s y = sy$ for any $y \in E$. Recall that, if $y \in E$, we denote by $\mathcal{F}_y$ the fiber of $\mathcal{F}$ over y. The flat connection on $\mathcal{F}$ gives a natural identification of the fibers $\mathcal{F}_y$ and $\mathcal{F}_{sy}$. Hence, the map $r_s : E \to E$ defines the "pull-back" map

$$r_s^* : \Omega^\bullet_{(2)}(E, \mathcal{F}) \to \Omega^\bullet_{(2)}(E, \mathcal{F}).$$

Note that r_s^* preserves the bigrading (2.3) and, hence, commutes with $\tau_{\sqrt{t}}$.

Lemma 2.8. *For any $t > 0, \alpha > 0$*

$$\nabla_{t,\alpha} = (\tau_{\sqrt{t}})^{-1} r^*_{\sqrt{t}} e^{-\alpha f} \nabla e^{\alpha f} (r^*_{\sqrt{t}})^{-1} \tau_{\sqrt{t}}. \tag{2.6}$$

Proof. Since $r^*_{\sqrt{t}}$ commutes with ∇ and with $\tau_{\sqrt{t}}$, it is enough to show that

$$r^*_{\sqrt{t}} \circ f \circ (r^*_{\sqrt{t}})^{-1} = tf, \tag{2.7}$$

where f is identified with the operator of multiplication by f. The equality (2.7) follows immediately from (2.4). □

Definition 2.9. The *Bismut Laplacian* $\Delta_{t,\alpha}$ of the bundle E associated to the metrics $g^E, h^{\mathcal{F}}$ is defined by the formula

$$\Delta_{t,\alpha} = \nabla_{t,\alpha} \nabla^*_{t,\alpha} + \nabla^*_{t,\alpha} \nabla_{t,\alpha}, \tag{2.8}$$

where $\nabla^*_{t,\alpha}$ denote the formal adjoint of $\nabla_{t,\alpha}$ with respect to the metrics $g^E, h^{\mathcal{F}}$. We denote by $\Delta^p_{t,\alpha}$ the restriction of $\Delta_{t,\alpha}$ on the space of p-forms.

Lemma 2.10. *For any $t > 0, \alpha > 0$*

$$\Delta_{t,\alpha} = (\tau_{\sqrt{t}})^{-1} r^*_{\sqrt{t}} \Delta_{1,\alpha} (r^*_{\sqrt{t}})^{-1} \tau_{\sqrt{t}}. \tag{2.9}$$

In particular, the operators $\Delta_{t,\alpha}$ and $\Delta_{1,\alpha}$ have the same spectrum.

Proof. Clearly, the operator $(\tau_{\sqrt{t}})^{-1} r^*_{\sqrt{t}}$ is orthogonal, i.e. the adjoint of $(\tau_{\sqrt{t}})^{-1} r^*_{\sqrt{t}}$ is equal to $(r^*_{\sqrt{t}})^{-1} \tau_{\sqrt{t}}$. The lemma follows now from Lemma 2.8. □

2.11. **The spectrum of $\Delta_{1,\alpha}$.** A simple calculation [CFKS, Proposition 11.13] shows that

$$\Delta_{1,\alpha} = \Delta + \alpha A + \alpha^2 |df|^2, \tag{2.10}$$

where $\Delta = \Delta_{1,0}$ is the usual Laplacian associated with the metrics $g^E, h^{\mathcal{F}}$ and A is a zero order operator.

Using the theory of globally elliptic differential operators [Sh1, Ch. IV], one easily obtains that, for $\alpha > 0$ large enough, $\Delta_{1,\alpha}$ has a discrete spectrum, and that the corresponding eigenspaces in $\Omega_{(2)}(E, \mathcal{F})$ have finite dimension.

The following theorem computes explicitly the cohomology of the deformed differential $\nabla_{1,\alpha}$ on the space of L_2-forms.

Theorem 2.12. *Let m denote the fiber dimension of E^- and let o denote the orientation bundle of E^-. If $\alpha > 0$ is large enough, then*

$$\dim \operatorname{Ker} \Delta^p_{1,\alpha} = \dim H^{p-m}(Z, \mathcal{F}_{|Z} \otimes o) \tag{2.11}$$

for any $p = 0, 1, \dots, n$.

The proof is given in the next section. In the case where $\mathcal{F}$ is a trivial line bundle, Theorem 2.12 was established by Bismut [B, Theorem 2.13]. Note that our proof is rather different from [B].

3. Proof of Theorem 2.12

In this section we use the notation of Section 2.

3.1. **A cohomological interpretation of** $\operatorname{Ker} \Delta_{1,\alpha}$**.** Assume that $\alpha > 0$ is sufficiently large so that the operator $\Delta_{1,\alpha}$ has a discrete spectrum (cf. Section 2.11).

Let $\widetilde{\Omega}^\bullet_{(2)}(E, \mathcal{F}; \nabla_{1,\alpha}) \imath \Omega^\bullet_{(2)}(E, \mathcal{F})$ be the space of smooth (C^∞) square integrable forms $\beta \in \Omega^\bullet_{(2)}(E, \mathcal{F})$ having the property $\nabla_{1,\alpha}\beta \in \Omega^\bullet_{(2)}(E, \mathcal{F})$. We denote by $H^\bullet_{(2)}(E, \mathcal{F}; \nabla_{1,\alpha})$ the cohomology of the complex

$$0 \to \widetilde{\Omega}^0_{(2)}(E, \mathcal{F}; \nabla_{1,\alpha}) \xrightarrow{\nabla_{1,\alpha}} \widetilde{\Omega}^1_{(2)}(E, \mathcal{F}; \nabla_{1,\alpha}) \xrightarrow{\nabla_{1,\alpha}} \cdots \xrightarrow{\nabla_{1,\alpha}} \widetilde{\Omega}^n_{(2)}(E, \mathcal{F}; \nabla_{1,\alpha}) \to 0.$$

One should think of $H^\bullet_{(2)}(E,\mathcal{F};\nabla_{1,\alpha})$ as of "deformed L_2 cohomology" of E with coefficients in $\mathcal{F}$ (recall from Section 2.6 that $\nabla_{1,\alpha} = e^{-\alpha f}\nabla e^{\alpha f} = \nabla + \alpha df$).

Lemma 3.2. *For any $p = 0, 1, \dots, n$, the following equality holds*

$$\dim \operatorname{Ker} \Delta^p_{1,\alpha} = \dim H^p_{(2)}(E,\mathcal{F};\nabla_{1,\alpha}). \tag{3.1}$$

Proof. Let $\widetilde{\nabla}_{1,\alpha}, \widetilde{\nabla}^*_{1,\alpha}$ denote the restriction of the operators $\nabla_{1,\alpha}, \nabla^*_{1,\alpha}$ on $\widetilde{\Omega}^\bullet_{(2)}(E,\mathcal{F};\nabla_{1,\alpha})$. To prove the lemma it is enough to show that the following decomposition holds

$$\widetilde{\Omega}^\bullet_{(2)}(E,\mathcal{F};\nabla_{1,\alpha}) = \operatorname{Ker}\Delta_{1,\alpha} \oplus \operatorname{Im}\widetilde{\nabla}_{1,\alpha} \oplus \operatorname{Im}\widetilde{\nabla}^*_{1,\alpha}. \tag{3.2}$$

Since the spectrum of $\Delta_{1,\alpha}$ is discrete, the space $\Omega^\bullet_{(2)}(E,\mathcal{F})$ decomposes into an orthogonal direct sum of closed subspaces

$$\Omega^\bullet_{(2)}(E,\mathcal{F}) = \operatorname{Ker}\Delta_{1,\alpha} \oplus \operatorname{Im}\Delta_{1,\alpha}.$$

Moreover, $\operatorname{Ker}\Delta_{1,\alpha} \subset \widetilde{\Omega}^\bullet_{(2)}(E,\mathcal{F};\nabla_{1,\alpha})$. Hence, any $\beta \in \widetilde{\Omega}^\bullet_{(2)}(E,\mathcal{F};\nabla_{1,\alpha})$ may be written as

$$\beta = \kappa + \Delta_{1,\alpha}\varphi = \kappa + \nabla_{1,\alpha}\nabla^*_{1,\alpha}\varphi + \nabla^*_{1,\alpha}\nabla_{1,\alpha}\varphi \tag{3.3}$$

where $\kappa \in \operatorname{Ker}\Delta_{1,\alpha}$ and $\varphi \in \Omega^\bullet_{(2)}(E,\mathcal{F})$. Then $\Delta_{1,\alpha}\varphi \in \widetilde{\Omega}^\bullet_{(2)}(E,\mathcal{F};\nabla_{1,\alpha})$.

Since $\Delta_{1,\alpha}$ is an elliptic operator, the form φ is smooth. Let $\langle\cdot,\cdot\rangle$ denote the L_2 scalar product on $\Omega^\bullet_{(2)}(E,\mathcal{F})$. For $\sigma \in \Omega^\bullet_{(2)}(E,\mathcal{F})$, we denote by $\|\sigma\|$ its norm with respect to this scalar product. Then

$$\|\nabla^*_{1,\alpha}\varphi\|^2 + \|\nabla_{1,\alpha}\varphi\|^2 = \langle\Delta_{1,\alpha}\varphi,\varphi\rangle < \infty.$$

Hence, $\nabla^*_{1,\alpha}\varphi, \nabla_{1,\alpha}\varphi \in \Omega^\bullet_{(2)}(E,\mathcal{F})$. Since the form φ is smooth, so are the forms $\nabla^*_{1,\alpha}\varphi, \nabla_{1,\alpha}\varphi$. Also, $\nabla_{1,\alpha}\nabla_{1,\alpha}\varphi = 0$ and, by (3.3), $\nabla_{1,\alpha}\nabla^*_{1,\alpha}\varphi \in \Omega^\bullet_{(2)}(E,\mathcal{F})$. Whence $\nabla^*_{1,\alpha}\varphi, \nabla_{1,\alpha}\varphi \in \widetilde{\Omega}^\bullet_{(2)}(E,\mathcal{F};\nabla_{1,\alpha})$. The decomposition (3.2) follows now from (3.3). □

In view of Lemma 3.2, to prove the Theorem 2.12 we only need to show that

$$\dim H^\bullet_{(2)}(E,\mathcal{F};\nabla_{1,\alpha}) = \dim H^{\bullet-m}(Z,\mathcal{F}_{|Z}\otimes o).$$

(Recall that m denote the dimension of fibers of E^-). In fact, we will prove that the complexes $\big(\widetilde{\Omega}^\bullet_{(2)}(E,\mathcal{F};\nabla_{1,\alpha}),\nabla_{1,\alpha}\big)$ and $\big(\Omega^\bullet(Z,\mathcal{F}_{|Z}),\nabla\big)$ are homotopy equivalent.

3.3. **The Thom isomorphism.** Let U^- be a *Thom form* of the bundle E^-, i.e. a closed *compactly supported* differential m-form on E^- with values in the orientation bundle o of E^- whose integrals over the fibers of E^- equal 1.

We denote by $\Omega_c^\bullet(E^-, \mathcal{F}_{|_{E^-}})$ the space of compactly supported differential forms on E^- with values in the restriction $\mathcal{F}_{|_{E^-}}$ of the bundle $\mathcal{F}$ on E^-. It is well known that the map

$$\Omega^\bullet(Z, \mathcal{F}_{|_Z} \otimes o) \to \Omega_c^{\bullet+m}(E^-, \mathcal{F}_{|_{E^-}}), \qquad \beta \mapsto U^- \wedge \beta, \tag{3.4}$$

induces the *Thom isomorphism*

$$H^\bullet(Z, \mathcal{F}_{|_Z} \otimes o) \to H_c^{\bullet+m}(E^-, \mathcal{F}_{|_{E^-}}) \tag{3.5}$$

from the cohomology of Z with coefficients in the bundle $\mathcal{F}_{|_Z} \otimes o$ onto the compactly supported cohomology of E^- with coefficients in $\mathcal{F}_{|_{E^-}}$.

Let $p : E^- \to Z$ denote the natural projection. Using the trivialization of $\mathcal{F}$ along the fibers of E^- determined by the flat connection ∇ one can define the push-forward map ("integration along the fibers")

$$p_* : \Omega_c^\bullet(E^-, \mathcal{F}_{|_{E^-}}) \to \Omega^{\bullet-m}(Z, \mathcal{F}_{|_Z} \otimes o).$$

The map p_* induces the map

$$H_c^\bullet(E^-, \mathcal{F}_{|_{E^-}}) \to H^{\bullet-m}(Z, \mathcal{F}_{|_Z} \otimes o),$$

which is inverse to the Thom isomorphism.

Remark 3.4. Note that the map p_* may be extended from $\Omega_c^\bullet(E^-, \mathcal{F}_{|_{E^-}})$ to a wider class of "rapidly decreasing forms". In particular, if $\beta \in \Omega^\bullet(E^-, \mathcal{F}_{|_{E^-}})$ is a square integrable form, then the form $p_*(e^{\alpha f}\beta)$ is well defined. This remark will be used later.

3.5. **Maps between $\widetilde{\Omega}_{(2)}^\bullet(E, \mathcal{F})$ and $\Omega^\bullet(Z, \mathcal{F}_{|_Z} \otimes o)$.** Let $j : E^- \to E$ be the inclusion and let

$$j^* : \Omega^\bullet(E, \mathcal{F}) \to \Omega^\bullet(E^-, \mathcal{F}_{|_{E^-}})$$

be the corresponding "pull-back" map. Set

$$\phi : \widetilde{\Omega}_{(2)}^\bullet(E, \mathcal{F}; \nabla_{1,\alpha}) \to \Omega^{\bullet-m}(Z, \mathcal{F}_{|_Z} \otimes o); \qquad \phi : \beta \mapsto p_* j^*(e^{\alpha f}\beta). \tag{3.6}$$

By Remark 3.4, this map is well defined. Clearly, $\phi\nabla_{1,\alpha} = \nabla'\phi$, where ∇' is the covariant derivative on the bundle $\mathcal{F}_{|_Z} \otimes o$. Hence, ϕ induces a map

$$\phi_* : H_{(2)}^\bullet(E, \mathcal{F}; \nabla_{1,\alpha}) \to H^{\bullet-m}(Z, \mathcal{F}_{|_Z} \otimes o). \tag{3.7}$$

Denote by $\pi : E \to E^-$ the natural projection and let

$$\pi^* : \Omega^\bullet(E^-, \mathcal{F}_{|_{E^-}}) \to \Omega^\bullet(E, \mathcal{F})$$

be the corresponding "pull-back" map defined by means of the connection ∇ (cf. Section 2.7). Set

$$\psi : \Omega^\bullet(Z, \mathcal{F}_{|Z} \otimes o) \to \widetilde{\Omega}^{\bullet+m}_{(2)}(E, \mathcal{F}; \nabla_{1,\alpha}); \qquad \psi : \delta \mapsto e^{-\alpha f}\pi^*(U^- \wedge p^*\delta). \tag{3.8}$$

Then $\psi\nabla' = \nabla_{1,\alpha}\psi$. Hence, ψ induces the map

$$\psi_* : H^\bullet(Z, \mathcal{F}_{|Z} \otimes o) \to H^{\bullet+m}_{(2)}(E, \mathcal{F}, \nabla_{1,\alpha}). \tag{3.9}$$

Proposition 3.6. *The maps ϕ_* and ψ_* are inverse of each other.*

Proof. Clearly, $\phi\psi = id$, and, hence, $\phi_*\psi_* = id$. We will show now that the map $\psi\phi$ is homotopic to id.

Consider the action of the group $\mathbb{R}^*$ of nonzero real numbers on $E = E^+ \oplus E^-$ defined by

$$h_t : y = (y^+, y^-) \mapsto (ty^+, y^-), \qquad t \in \mathbb{R}^*. \tag{3.10}$$

As in Section 2.7, the flat connection on $\mathcal{F}$ defines naturally the pull-back map

$$h_t^* : \widetilde{\Omega}^\bullet_{(2)}(E, \mathcal{F}; \nabla_{1,\alpha}) \to \widetilde{\Omega}^\bullet_{(2)}(E, \mathcal{F}; \nabla_{1,\alpha})$$

associated with h_t.

Assume that $\mathcal{R}$ is the vector field on E generating the action (3.10) (the *Euler vector field* in the direction E^+). Let $\iota(\mathcal{R})$ denote the interior multiplication by $\mathcal{R}$. If

$$\mathcal{L}(\mathcal{R}) = \nabla\iota(\mathcal{R}) + \iota(\mathcal{R})\nabla \tag{3.11}$$

denote the *Lie derivative* along $\mathcal{R}$, then

$$\frac{d}{dt}h_t^*(\beta) = h_t^*(\mathcal{L}(\mathcal{R})\beta)t^{-1}. \tag{3.12}$$

Define a map $H : \widetilde{\Omega}^\bullet_{(2)}(E, \mathcal{F}; \nabla_{1,\alpha}) \to \widetilde{\Omega}^{\bullet-1}_{(2)}(E, \mathcal{F}; \nabla_{1,\alpha})$ by

$$H\beta = \int_0^1 h_t^*(\iota(\mathcal{R})\beta)t^{-1}dt, \qquad \beta \in \widetilde{\Omega}^\bullet_{(2)}(E, \mathcal{F}; \nabla_{1,\alpha}), \tag{3.13}$$

and let $\widetilde{H} = e^{-\alpha f}He^{\alpha f}$. Note that the integral in the definition of H converges because $\mathcal{R}$ vanishes at E^-.

Lemma 3.7. *The following homotopy formula holds*

$$\beta - \psi(\phi\beta) = (\nabla_{1,\alpha}\widetilde{H} + \widetilde{H}\nabla_{1,\alpha})\beta, \quad \textit{for all} \quad \beta \in \widetilde{\Omega}^\bullet_{(2)}(E, \mathcal{F}; \nabla_{1,\alpha}). \tag{3.14}$$

Proof. If $\beta \in \widetilde{\Omega}^{\bullet}_{(2)}(E,\mathcal{F};\nabla_{1,\alpha})$, let $\beta_t = h_t^*\beta$, and observe that $\beta_0 = e^{\alpha f}\psi(\phi e^{-\alpha f}\beta)$ and $\beta_1 = \beta$. Differentiating by t, using (3.11),(3.12) and integrating we get

$$\beta - e^{\alpha f}\psi(\phi e^{-\alpha f}\beta) = (\nabla H + H\nabla)\beta, \qquad \text{for all} \qquad \beta \in \widetilde{\Omega}^{\bullet}_{(2)}(E,\mathcal{F};\nabla_{1,\alpha}).$$

Using $\nabla_{1,\alpha} = e^{-\alpha f}\nabla e^{\alpha f}$, we obtain the lemma. □

Lemma 3.7, combined with the relation $\phi\psi = id$, completes the proof of Proposition 3.6 and, hence, of Theorem 2.12.

4. The Bismut Laplacian on a manifold

Let ω be a closed real valued differential 1-form on a compact manifold M, which is non-degenerate in the sense of Bott (cf. Section 0.2). In this section we construct a two-parameter family $\Delta_{t,\alpha}$ of Laplacians acting on the space $\Omega^{\bullet}(M,\mathcal{F})$ of differential forms on a compact manifold M with values in a flat complex vector bundle $\mathcal{F}$. This construction is similar to the construction of Bismut [B]. Next we show that, for large values of the parameter t, the $\dim \operatorname{Ker} \Delta_{t,\alpha}$ is equal to the number of "small" eigenvalues of the Bismut Laplacian acting on the normal bundle to the zero set of ω, where the normal bundle is considered as a non-compact manifold.

4.1. **The Bismut Laplacian on the normal bundle.** Throughout this section we use the notations introduced in Section 0.2. In particular, C denotes the set of critical points of ω, i.e. the subset of M on which ω vanishes. Recall that the form ω is assumed to be non-degenerate in the sense of Bott, i.e. C is a union of disjoint compact connected manifolds and the Hessian of ω is a non-degenerate quadratic form on the normal bundle $\nu(C)$ to C in M.

As we have mentioned in Section 0.2, the bundle $\nu(C)$ splits into the Whitney sum of two subbundles

$$\nu(C) = \nu^+(C) \oplus \nu^-(C) \tag{4.1}$$

such that the Hessian of ω is strictly positive on $\nu^+(C)$ and strictly negative on $\nu^-(C)$.

Denote by $p : \nu(C) \to C$ the natural projection and by $\mathcal{F}_{|C}$ the restriction of the bundle $\mathcal{F}$ on C. Then the pull-back $p^*\mathcal{F}_{|C}$ is a flat vector bundle over $\nu(C)$ which will be denoted by $\mathcal{F}_\nu$.

For each connected component Z of C we denote by $\nu(Z) = \nu(C)_{|Z}$ the normal bundle to Z in M and by $\mathcal{F}_Z$ the restriction of the bundle $\mathcal{F}_\nu$ on $\nu(Z)$. Let $g^{\nu(Z)}$ be a Riemannian metric on the non-compact manifold $\nu(Z)$ constructed as explained in Section 2.3, starting from a Riemannian metric on Z, an Euclidean metric on $\nu(C)$ and an Euclidean connection on $\nu(C)$. Let $h^{\mathcal{F}_Z}$ be a Hermitian metric on $\mathcal{F}_Z$ constructed as explained in Section 2.4. We denote by $\Delta^Z_{t,\alpha}$ the Bismut Laplacian associated with the metrics $g^{\nu(Z)}, h^{\mathcal{F}_Z}$ (cf. Definition 2.9).

We denote by $g^{\nu(C)}$ the Riemannian metric on $\nu(C)$ induced by the metrics $g^{\nu(Z)}$ and by $h^{\mathcal{F}_\nu}$ the Hermitian metric on $\mathcal{F}_\nu$ induced by the metrics $h^{\mathcal{F}_Z}$.

Let

$$\Delta^C_{t,\alpha} = \bigoplus_Z \Delta^Z_{t,\alpha} : \Omega^\bullet\big(\nu(C), \mathcal{F}_\nu\big) \to \Omega^\bullet\big(\nu(C), \mathcal{F}_\nu\big)$$

(the sum is taken over all connected components of C) be the operator whose restriction on $\Omega^\bullet\big(\nu(Z), \mathcal{F}_Z\big)$ equals $\Delta^Z_{t,\alpha}$. Clearly,

$$\operatorname{Ker} \Delta^C_{t,\alpha} = \bigoplus_Z \operatorname{Ker} \Delta^Z_{t,\alpha}.$$

4.2. **Metrics on M and $\mathcal{F}$.** By the generalized Morse lemma [H, Ch. 6] there exist a neighborhood U of the zero section in $\nu(C)$ and an embedding $\psi : U \to M$ such that the restriction of ψ on C is the identity map and if $y = (y^+, y^-) \in U$, then

$$\big(f \circ \psi\big)(y) = \frac{|y^+|^2}{2} - \frac{|y^-|^2}{2}. \tag{4.2}$$

In the sequel, we will identify U and $\psi(U)$. In particular, we will consider $g^{\nu(C)}$ as a metric on $\psi(U)$ and $h^{\mathcal{F}_\nu}$ as a metric on the restriction $\mathcal{F}_{|U}$ of $\mathcal{F}$ on U. Let g^M be any Riemannian metric on M whose restriction on U equals $g^{\nu(C)}$ and let $h^{\mathcal{F}}$ be any Hermitian metric on $\mathcal{F}$ whose restriction on U equals $h^{\mathcal{F}_\nu}$.

4.3. **The Bismut Laplacian on M.** Let V be a neighborhood of C in $\nu(C)$ whose closure is contained in U. Fix a function $\phi \in C^\infty(\nu(C))$ such that $0 \le \phi \le 1$, $\phi(x) = 1$ if $x \in V$ and $\phi(x) = 0$ if $x \notin U$.

Recall that the map $\tau_{\sqrt{t}}$ was defined in Section 2.5. Using the identification $\psi : U \to \psi(U)$ we can consider ϕ as a function on M and $\phi\tau_{\sqrt{t}}$ as an operator on $\Omega^\bullet(M, \mathcal{F})$.

For any $t > 0, \alpha > 0$, we define a new differential $\nabla_{t,\alpha}$ on $\Omega^\bullet(M, \mathcal{F})$ by the formula

$$\nabla_{t,\alpha} = (\phi\tau_{\sqrt{t}} + (1-\phi))^{-1}(\nabla + t\alpha\omega)(\phi\tau_{\sqrt{t}} + (1-\phi)). \tag{4.3}$$

Here $\nabla + t\alpha\omega : \Omega^\bullet(M, \mathcal{F}) \to \Omega^{\bullet+1}(M, \mathcal{F})$ denotes the map

$$\theta \mapsto \nabla\theta + t\alpha\omega \wedge \theta, \qquad \theta \in \Omega^\bullet(M, \mathcal{F}).$$

Remark 4.4. Our definition of $\nabla_{t,\alpha}$ is slightly different from [B] even in the case when ω is cohomologious to 0 and $\mathcal{F}$ is a trivial line bundle. However this difference does not change the asymptotic behavior of the spectrum of the corresponding Laplacian (cf. Definition 4.6).

Remark 4.5. Note that on V the formulae (4.3) and (2.5) coincide. Also on $M\backslash U$ we have $\nabla_{t,\alpha} = \nabla + t\alpha\omega$.

Definition 4.6. For $t > 0, \alpha > 0$, the *Bismut Laplacian* on M is the operator

$$\Delta_{t,\alpha} = \nabla_{t,\alpha}\nabla^*_{t,\alpha} + \nabla^*_{t,\alpha}\nabla_{t,\alpha} : \Omega^\bullet(M,\mathcal{F}) \to \Omega^\bullet(M,\mathcal{F}), \tag{4.4}$$

where $\nabla^*_{t,\alpha}$ is the formal adjoint of $\nabla_{t,\alpha}$ with respect to the metrics $g^M, h^{\mathcal{F}}$. We denote by $\Delta^p_{t,\alpha}$ the restriction of $\Delta_{t,\alpha}$ on $\Omega^p(M,\mathcal{F})$.

4.7. We now fix a number $\alpha > 0$ large enough, so that the spectrum of $\Delta^C_{1,\alpha}$ is discrete and the equality (2.11) holds.

Let A be a self-adjoint operator with discrete spectrum. For any $\lambda > 0$, we denote by $N(\lambda, A)$ the number of the eigenvalues of A not exceeding λ (counting multiplicity).

The following theorem plays a central role in our proof of Theorem 0.3.

Theorem 4.8. *Let λ_p $(p = 0, 1, \dots, n)$ be the smallest non-zero eigenvalue of $\Delta^{C,p}_{1,\alpha}$. Then for any $\varepsilon > 0$ there exists $T > 0$ such that for all $t > T$*

$$N(\lambda_p - \varepsilon, \Delta^p_{t,\alpha}) = \dim \operatorname{Ker} \Delta^{C,p}_{1,\alpha}. \tag{4.5}$$

The rest of this section is occupied with the proof of Theorem 4.8.

4.9. **Estimate from above on $N(\lambda_p - \varepsilon, \Delta^p_{t,\alpha})$.** We will first show that

$$N(\lambda_p - \varepsilon, \Delta^p_{t,\alpha}) \le \dim \operatorname{Ker} \Delta^{C,p}_{1,\alpha}. \tag{4.6}$$

To this end we will estimate the operator $\Delta^p_{t,\alpha}$ from below. We will use the technique of [Sh2], adding some necessary modifications.

Recall from Section 4.2 that U is a tubular neighborhood of the zero section in $\nu(C)$ and that we have fixed an embedding $\psi : U \to M$. Also in Section 4.3 we have chosen $V \subset U$.

For $x \in U$, we denote by $|x|$ its norm with respect to the fixed Euclidean structure on $\nu(C)$.

There exists $\kappa > 0$ such that the set $\{x \in U : |x| < 2\kappa\}$ is contained in V. Let us fix a C^∞ function $j : [0, +\infty) \to [0, 1]$ such that $j(s) = 1$

for $s \le \kappa$, $j(s) = 0$ for $s \ge 2\kappa$ and the function $(1-j^2)^{1/2}$ is C^∞. We define functions $J, \overline{J} \in C^\infty(\nu(C))$ by

$$J(x) = j(|x|); \qquad \overline{J}(x) = \big(1 - j(|x|)^2\big)^{\frac{1}{2}}.$$

Using the diffeomorphism $\psi : U \to \psi(U)$ we can and we will consider $J, \overline{J}$ as functions on M.

We identify the functions $J, \overline{J}$ with the corresponding multiplication operators. For operators A, B, we denote by $[A,B] = AB - BA$ their commutator.

The following version of IMS localization formula (cf. [CFKS]) is due to Shubin [Sh2, Lemma 3.1].

Lemma 4.10. *The following operator identity holds*

$$\Delta^p_{t,\alpha} = \overline{J}\Delta^p_{t,\alpha}\overline{J} + J\Delta^p_{t,\alpha}J + \frac{1}{2}[\overline{J},[\overline{J},\Delta^p_{t,\alpha}]] + \frac{1}{2}[J,[J,\Delta^p_{t,\alpha}]]. \tag{4.7}$$

Proof. Using the equality $J^2 + \overline{J}^2 = 1$ we can write

$$\Delta^p_{t,\alpha} = J^2\Delta^p_{t,\alpha} + \overline{J}^2\Delta^p_{t,\alpha} = J\Delta^p_{t,\alpha}J + \overline{J}\Delta^p_{t,\alpha}\overline{J} + J[J,\Delta^p_{t,\alpha}] + \overline{J}[\overline{J},\Delta^p_{t,\alpha}].$$

Similarly,

$$\Delta^p_{t,\alpha} = \Delta^p_{t,\alpha}J^2 + \Delta^p_{t,\alpha}\overline{J}^2 = J\Delta^p_{t,\alpha}J + \overline{J}\Delta^p_{t,\alpha}\overline{J} - [J,\Delta^p_{t,\alpha}]J - [\overline{J},\Delta^p_{t,\alpha}]\overline{J}.$$

Summing these identities and dividing by 2 we come to (4.7). □

We will now estimate each one of the summands in the right hand side of (4.7).

Lemma 4.11. *There exist $c > 0$, $T > 0$ such that, for any $t > T$,*

$$\overline{J}\Delta^p_{t,\alpha}\overline{J} \ge ct\overline{J}^2 I. \tag{4.8}$$

Proof. Let $\langle\cdot,\cdot\rangle$ be the natural L_2 scalar product on $\Omega^\bullet(M,\mathcal{F})$ determined by the metrics $g^M, h^{\mathcal{F}}$. For $x \in \Omega^\bullet(M,\mathcal{F})$, we denote by $\|x\|$ its norm with respect to this scalar product.

We can assume that $T > 1$. Using (4.3), one easily checks that for any $a \in \Omega^p(M,\mathcal{F})$ and any $t > T > 1$

$$\|\nabla_{t,\alpha}a\|^2 \ge \frac{1}{t}\|(\nabla + t\alpha\omega)a\|^2, \qquad \|\nabla^*_{t,\alpha}a\|^2 \ge \frac{1}{t}\|(\nabla + t\alpha\omega)^*a\|^2.$$

Hence,

$$\begin{aligned}\langle\overline{J}\Delta^p_{t,\alpha}\overline{J}a, a\rangle &= \|\nabla_{t,\alpha}\overline{J}a\|^2 + \|\nabla^*_{t,\alpha}\overline{J}a\|^2 \\ &\ge \frac{1}{t}\|(\nabla + t\alpha\omega)\overline{J}a\|^2 + \frac{1}{t}\|(\nabla + t\alpha\omega)^*\overline{J}a\|^2 = \frac{1}{t}\langle\overline{J}\Delta^p_{1,\alpha t}\overline{J}a, a\rangle.\end{aligned} \tag{4.9}$$

A simple calculation (cf. [CFKS, Proposition 11.13]) shows that

$$\Delta_{1,t\alpha} = \Delta + \alpha t A + \alpha^2 t^2 |\omega|^2, \tag{4.10}$$

where A is a zero order differential operator and $\Delta = \nabla\nabla^* + \nabla^*\nabla$ is the undeformed Laplacian associated with the metrics $g^M, h^{\mathcal{F}}$.

From (4.10) and (4.9), we get

$$\overline{J}\Delta^p_{t,\alpha}\overline{J} \geq \frac{1}{t}\overline{J}\Delta^p_{1,\alpha t}\overline{J} = \frac{1}{t}\overline{J}\Delta^p\overline{J} + \alpha\overline{J}A\overline{J} + t\alpha^2\overline{J}|\omega|^2\overline{J}. \tag{4.11}$$

Since A is a zero order operator, there exists $M > 0$ such that $A > -M$. Also $\Delta^p \geq 0$. Taking

$$c = \frac{\alpha^2}{2}\min_{x\in\operatorname{supp}\overline{J}}|\omega|^2, \qquad T = \frac{M\alpha}{c},$$

and using (4.11) we get (4.8) □

Let $P^p_{t,\alpha} : \Omega_{(2)}\big(\nu^-(C), \mathcal{F}_\nu\big) \to \operatorname{Ker}\Delta^{C,p}_{t,\alpha}$ be the orthogonal projection. This is a finite rank operator on $\Omega_{(2)}\big(\nu^-(C), \mathcal{F}_\nu\big)$ and its rank equals $\dim\operatorname{Ker}\Delta^{C,p}_{1,\alpha}$. Clearly,

$$\Delta^{C,p}_{t,\alpha} + P^p_{t,\alpha} \geq \lambda_p I. \tag{4.12}$$

Using the identification $\psi : U \to \psi(U)$ we can consider $JP^p_{t,\alpha}J$ and $J\Delta^{C,p}_{t,\alpha}J$ as operators on $\Omega^p(M,\mathcal{F})$. It follows from Remark 4.5 that $J\Delta^p_{t,\alpha}J = J\Delta^{C,p}_{t,\alpha}J$. Hence, (4.12) implies the following

Lemma 4.12. *For any $t > 0$*

$$J\Delta^p_{t,\alpha}J + JP^p_{t,\alpha}J \geq \lambda_p J^2 I, \qquad \operatorname{rk} JP^p_{t,\alpha}J \leq \dim\operatorname{Ker}\Delta^{C,p}_{1,\alpha}. \tag{4.13}$$

For an operator $A : \Omega^p(M,\mathcal{F}) \to \Omega^p(M,\mathcal{F})$, we denote by $\|A\|$ its norm with respect to L_2 scalar product on $\Omega^p(M,\mathcal{F})$.

Lemma 4.13. *There exists $C > 0$ such that*

$$\|[J,[J,\Delta^p_{t,\alpha}]]\| \leq Ct^{-1} \tag{4.14}$$

for any $t > 0, \alpha > 0$.

Proof. The left hand side of (4.14) is a zero order operator supported on V. Hence, it is enough to estimate $[J,[J,\Delta^{C,p}_{t,\alpha}]]$. Since

$$(r^*_{\sqrt{t}})^{-1}J(x)r^*_{\sqrt{t}} = J(xt^{-\frac{1}{2}}), \qquad (\tau_{\sqrt{t}})J(x)(\tau_{\sqrt{t}})^{-1} = J(x),$$

Lemma 2.10 implies

$$[J,[J,\Delta^{C,p}_{t,\alpha}]] = (\tau_{\sqrt{t}})^{-1} r^*_{\sqrt{t}} [J(xt^{-\frac{1}{2}}),[J(xt^{-\frac{1}{2}}),\Delta^{C,p}_{1,\alpha}]] (r^*_{\sqrt{t}})^{-1}\tau_{\sqrt{t}}. \tag{4.15}$$

Since the operators $(\tau_{\sqrt{t}})^{-1} r^*_{\sqrt{t}}$ and $(r^*_{\sqrt{t}})^{-1}\tau_{\sqrt{t}}$ are mutually adjoint, (4.15) implies

$$\|[J,[J,\Delta^{C,p}_{t,\alpha}]]\| = \|[J(xt^{-\frac{1}{2}}),[J(xt^{-\frac{1}{2}}),\Delta^{C,p}_{1,\alpha}]]\|.$$

Recall that $g^{\nu(C)}$ denotes the Riemannian metric on $\nu(C)$. Suppose that we have chosen local coordinates $(x^1,\dots,x^n)$ near a point $x\in\nu(C)$. Set

$$g_{i,j} = g^{\nu(C)}(\partial/\partial x^i, \partial/\partial x^j), \qquad 1\le i,j\le n.$$

Using (2.10), we see that

$$\Delta^{C,p}_{1,\alpha} = -\sum_{i,j=1}^n g_{i,j}\frac{\partial^2}{\partial x^i\partial x^j} + B,$$

where B is an order 1 differential operator. Hence,

$$[J(xt^{-\frac{1}{2}}),[J(xt^{-\frac{1}{2}}),\Delta^{C,p}_{1,\alpha}]] = -2\sum_{i,j=1}^n g_{i,j}\frac{\partial J(xt^{-\frac{1}{2}})}{\partial x^i}\frac{\partial J(xt^{-\frac{1}{2}})}{\partial x^j}.$$

Since the derivatives of $J(xt^{-\frac{1}{2}}) = j(|x|t^{-\frac{1}{2}})$ can be estimated as $O(t^{-\frac{1}{2}})$ the inequality (4.14) follows immediately. $\square$

Similarly, one shows that

$$\|[\overline{J},[\overline{J},\Delta^p_{t,\alpha}]]\| \le Ct^{-1}. \tag{4.16}$$

¿From Lemma 4.10, Lemma 4.11, Lemma 4.12, Lemma 4.13 and (4.16) we get the following

Corollary 4.14. *For any $\varepsilon>0$, there exists $T>0$ such that for any $t>T$*

$$\Delta^p_{t,\alpha} + JP^p_{t,\alpha}J \ge (\lambda_p-\varepsilon)I, \qquad \operatorname{rk} JP^p_{t,\alpha}J \le \dim\operatorname{Ker}\Delta^{C,p}_{1,\alpha}. \tag{4.17}$$

The estimate (4.6) follows now from Corollary 4.14 and the following general lemma [RS, p. 270].

Lemma 4.15. *Assume that A, B are self-adjoint operators in a Hilbert space $\mathcal{H}$ such that $\operatorname{rk} B\le k$ and there exists $\mu>0$ such that*

$$\langle (A+B)u,u\rangle \ge \mu\langle u,u\rangle \quad \textit{for any} \quad u\in \operatorname{Dom}(A).$$

Then $N(\mu-\varepsilon, A)\le k$ for any $\varepsilon>0$.

4.16. **Estimate from below on $N(\lambda_p-\varepsilon,\Delta^p_{t,\alpha})$.** To prove Theorem 4.8 it remains to show now that

$$N(\lambda_p-\varepsilon,\Delta^p_{t,\alpha})\ge \dim\operatorname{Ker}\Delta^{C,p}_{1,\alpha}. \tag{4.18}$$

Let $E^p_{t,\alpha}$ be the subspace of $\Omega^p(M,\mathcal{F})$ spanned by the eigenvectors of $\Delta^p_{t,\alpha}$ corresponding to the eigenvalues $\lambda\le\lambda_p-\varepsilon$ and let $\Pi^p_{t,\alpha}:\Omega^p(M,\mathcal{F})\to E^p_{t,\alpha}$ be the orthogonal projection. Then

$$\operatorname{rk}\Pi^p_{t,\alpha}=N(\lambda_p-\varepsilon,\Delta^p_{t,\alpha}). \tag{4.19}$$

Using the diffeomorphism $\psi:U\to\psi(U)$ we can consider $J\Pi^p_{t,\alpha}J$ as an operator on $\Omega_{(2)}\big(\nu^-(C),\mathcal{F}_\nu\big)$. The proof of the following lemma does not differ from the proof of Corollary 4.14.

Lemma 4.17. *For any $\delta>\varepsilon$, there exists $T>0$ such that for any $t>T$*

$$\Delta^{C,p}_{t,\alpha}+J\Pi^p_{t,\alpha}J\ge(\lambda_p-\delta)I. \tag{4.20}$$

The estimate (4.18) follows now from (4.19), Lemma 4.17 and Lemma 4.15.

5. Proof of Theorem 0.3

Recall that the number $\alpha>0$ was fixed in Section 4.7. We will use the notation of Section 4. In particular, λ_p $(p=0,1,2,\dots)$ denotes the smallest non-zero eigenvalues of $\Delta^{C,p}_{1,\alpha}$. By Theorem 4.8 and Lemma 1.3, we can choose $t>0$ such that

$$N(\lambda_p/2,\Delta^p_{t,\alpha})=\dim\operatorname{Ker}\Delta^{C,p}_{1,\alpha},\qquad p=0,1,\dots,n, \tag{5.1}$$

and the Novikov numbers $\beta_p(\xi,\mathcal{F})$ are equal to the dimension of the cohomology of the de Rham complex $\Omega^\bullet(M,\mathcal{F})$ with the deformed differential

$$\nabla_t\theta=\nabla\theta+t\omega\wedge\theta,\qquad \theta\in\Omega^\bullet(M,\mathcal{F}).$$

The latter complex is isomorphic to the complex

$$0\to\Omega^0(M,\mathcal{F})\xrightarrow{\nabla_{t,\alpha}}\Omega^1(M,\mathcal{F})\xrightarrow{\nabla_{t,\alpha}}\cdots\xrightarrow{\nabla_{t,\alpha}}\Omega^n(M,\mathcal{F})\to 0, \tag{5.2}$$

where $\nabla_{t,\alpha}$ is defined by (4.3).

Let $E^p_{t,\alpha}$ $(p=0,1,\dots,n)$ be the subspace of $\Omega^p(M,\mathcal{F})$ spanned by the eigenvectors of $\Delta^p_{t,\alpha}$ corresponding to the eigenvalues $\lambda\le\lambda_p/2$. From (5.1) and Theorem 2.12, we obtain

$$\dim E^p_{t,\alpha}=\sum_Z \dim H^{p-\operatorname{ind}(Z)}(Z,\mathcal{F}_{|Z}\otimes o(Z)), \tag{5.3}$$

where the sum ranges over all connected components Z of the set C of critical points of ω.

Since the operator $\Delta_{t,\alpha}$ commutes with $\nabla_{t,\alpha}$, the pair $(E^\bullet_{t,\alpha}, \nabla_{t,\alpha})$ is a subcomplex of (5.2) and the inclusion induces an isomorphism of cohomology

$$H^\bullet(E^\bullet_{t,\alpha}, \nabla_{t,\alpha}) = H^\bullet(\Omega^\bullet(M, \mathcal{F}), \nabla_{t,\alpha}) = H^\bullet(M, \rho_t \otimes \mathcal{F}).$$

Hence,

$$\dim H^p(E^\bullet_{t,\alpha}, \nabla_{t,\alpha}) = \beta_p(\xi, \mathcal{F}), \qquad p = 0, 1, \dots, n. \tag{5.4}$$

Theorem 0.3 follows now from (5.3), (5.4) by standard arguments (cf. [Bo2]).

References

[BGV] N. Berline, E. Getzler, M. Vergne, *Heat Kernels and Dirac Operators*, Springer-Verlag,1992.

[B] J.-M. Bismut, *The Witten complex and the degenerate Morse inequalities*, J. Dif. Geom. **23** (1986) 207–240.

[Bo1] R. Bott, *Non degenerate critical manifolds*, Ann. of Math. **60** (1954) 248–261.

[Bo2] R. Bott, *Morse theory indomitable*, Publ. Math. IHES **68** (1988), 99–114.

[BF] M. Braverman, M.S. Farber, *The Novikov-Bott inequalities*, to appear in C.R. Acad. Sci. Paris.

[CFKS] H.L. Cycon, R.G. Froese, W. Kirsch, B. Simon, *Schrödinger operators with applications to quantum mechanics and global geometry*, Texts and Monographs in Physics, Springer-Verlag, 1987.

[F1] M.S. Farber, *Exactness of the Novikov inequalities*, Functional Anal. Appl. **19** (1985) 40–48.

[F2] M.S. Farber, *Singularities of analytic torsion*, J. of Diff. Geom. **41** (1995) 520–572.

[FL] M.S. Farber, J.P. Levine, *Jumps of the eta-invariant*, to appear in Math. Zeitschrift.

[Ha] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, 1977.

[HS1] B. Helffer, J. Sjöstrand, *Puits multiples en limite semi-classicue. IV. Étude du complexe de Witten*, Comm. in P.D.E. **10** (1985) 245–340.

[HS2] ———, *Puits multiples en limite semi-classicue. VI. (Cas des puits sous-variétés)*, Ann. Inst. Henri Poincaré, **46** (1987) 353–372.

[HS3] ———, *A proof of the Bott inequalities*, in Algebraic Analysis, Vol I, M. Kashiwara, T. Kawai ed., New-York, Academic Press, 1988, 171–183.

[H] M. Hirsch, *Differential topology*, Graduate Texts in Math., No. 33, Springer, Berlin, 1976.

[N1] S.P. Novikov, *Multivalued functions and functionals. An analogue of the Morse theory*, Soviet Math. Dokl. **24** (1981) 222–226.

[N2] ———, *The Hamiltonian formalism and a multivalued analogue of Morse theory*, Russian Math. Surveys, **37** (1982) 1–56.

[N3] ———, *Bloch homology, critical points of functions and closed 1-forms*, Soviet Math. Dokl. **33** (1986) 551–555.

[P1] A. Pazhitnov, *An analytic proof of the real part of Novikov's inequalities*, Soviet Math. Dokl. **35** (1987) 456–457.

[P2] ———, *Morse theory of closed 1-forms*, Lecture Notes in Math., **1474**, Springer, Berlin, 1991.

[RS] M. Reed, B. Simon, *Methods of modern mathematical physics IV: Analysis of operators*, Academic Press, London, 1978.

[R] D. Rolfsen, *Knots and Links*, Publish or Perish, Inc., 1976.

[Sh1] M.A. Shubin, *Pseudodifferential operators and spectral theory*, Springer-Verlag, 1987.

[Sh2] ———, *Semiclassical asymptotics on covering manifolds and Morse inequalities*, Preprint, 1994.

[W] E. Witten, *Supersymmetry and Morse Theory*, J. of Diff. Geom. **17** (1982) 661–692

School of Mathematical Sciences, Tel-Aviv University, Ramat-Aviv 69978, Israel

E-mail address: maxim@math.tau.ac.il, farber@math.tau.ac.il