NORTHEASTERN UNIVERSITY
Department of Mathematics Fall '03

Conic Sections and Quadratic Surface Lab

Goals

By the end of this lab you should:

1.) Be familar with the important features of ellipses and hyperbolas. For ellipses these are the semi-major and semi-minor axes, for hyperbolas these are the vertices and asymptotes.

2.) Understand the connection between the equation of the ellipse, and the length of the semi-major and semi-minor axes.

3.) Understand the connection between the equation of the hyperbola and its vertices and asymptotes.

4.) Be able to draw ellipses and hyperbolas whose equations are in standard form.

5) Be able to recognize the kind of quadratic surface defined by an equation from the graph of the equation.

Introduction

Calculus is about functions. Some basic examples are linear functions like $f(x, y) = 3x + 4y$ and quadratic functions like $g(x, y) = 3x^2 + 4y^2$ or $h(x, y, z) = 3x^2 + 4y^2 - z^2$. Calculus is powerful, because it often reduces understanding a complicated function to understanding a linear or quadratic one. Still, we have to understand the linear and quadratic functions. In order to understand a quadratic function like $g(x, y) = 3x^2 + 4y^2$, we have to understand its levels. These are the curves $g(x, y) = c$, and they are examples of ellipses.

In the first two parts of this computer lab, we will review ellipses and hyperbolas. We will need to be able to draw these curves so that we can draw level curves of quadratic functions of two variables. It will turn out that the pattern of level curves around the critical points of a function of two variables in "most" cases looks like a family of concentric ellipses or hyperbolas.

Specifically, in this lab you will examine the equations of ellipses and hyperbolas in standard form, see how the shape of the conic section is related to the terms of the equation, and review how to draw the graphs of the equations.

This should give you a good feel for quadratic functions in two variables. As a start to understanding quadratic functions in three variables, in the third part of the lab, we give you some practice recognizing some of the level sets of quadratic functions of three variables.

This lab will also introduce you to the MAPLE software package, which is an extremely powerful tool for doing mathematical calculations and graphing.

Background

As we all know, the equation $x^2 + y^2 - 1 = 0$ describes a circle in the xy-plane. In general, quadratic equations of the form

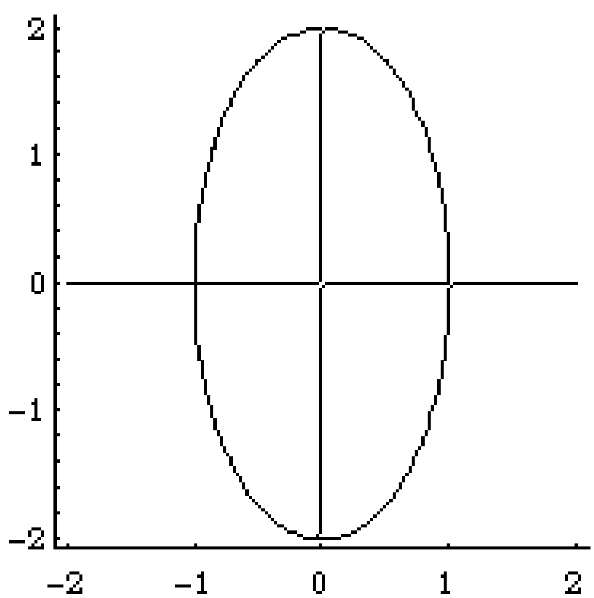
$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

describe plane curves known as **conic sections**. For different choices of the constants A, B, C, D, E, F you can get an ellipse, hyperbola, parabola, pair of lines, a single line or a point as the graph of the equation. These sets are called conic sections because they are the sets you can get if you intersect a cone with a plane.

Ellipses

The graph of the equation $x^2/a^2 + y^2/b^2 = 1$ is an **ellipse in standard form**. An ellipse has two perpendicular lines of symmetry; for an ellipse in standard form these lines of symmetry are the x and y axes. (We say a line L is a line of symmetry for a shape if L divides the shape into 2 congruent pieces, and the two pieces match if we rotate one around L.) Every ellipse has a **major axis** and a **semi-major axis**, a **minor axis** and a **semi-minor axis**.

For an ellipse in standard form look at the part of the coordinate axes lying inside the ellipse; the longer of the two segments is the major axis and the shorter is the minor axis. *The part of the major axis which runs from the center of the ellipse to the end of the major axis is called the semi-major axis*, while *the part of the minor axis from the center of the ellipse to the end of the minor axis is the semi-minor axis.* In the graph below, the semi-major axis lies on the y-axis and has length 2, while the semi-minor axis lies on the x-axis and has length 1.



Hyperbolas

The graph of the equation $x^2/a^2 - y^2/b^2 = 1$ is a **hyperbola in standard form**. The x and y axes are lines of symmetry for this shape. *The hyperbola has two points which are* *closest to the origin*; these are called **vertices**, and they lie on the x-axis, if the hyperbola is in standard form.

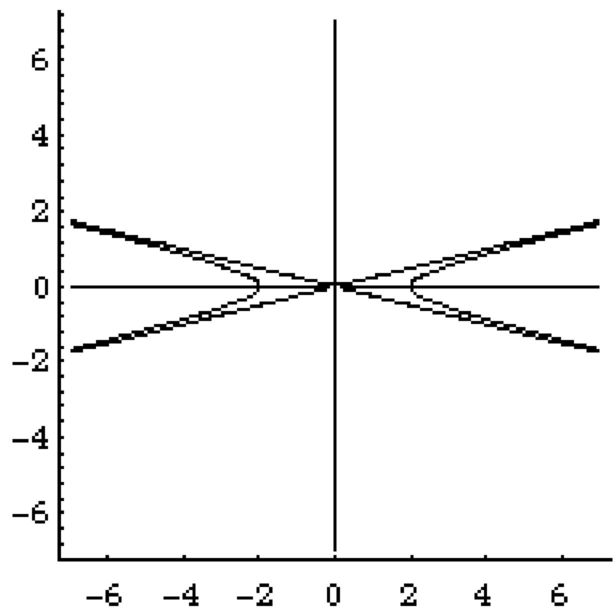
The hyperbola also has two **asymptotes**. *The equations for the asymptotes are gotten by taking the quadratic function* $x^2/a^2 - y^2/b^2$*, setting it equal to zero, factoring it, and setting each factor equal to zero.* The two equations for the asymptotes that we get are:

$$x/a + y/b = 0$$

and

$$x/a - y/b = 0$$

The figure below is a hyperbola, with vertices at (-2,0), (2,0) and with asymptotes $4y = x$, $-4y = x$. So that you can see how closely the hyperbola hugs the asymptotes, we have included them in the figure.



Ellipses

Question 1.

(a) Plot $x^2/a^2 + y^2 = 1$ for $a = 1, 3, 5, 7$. We suggest you use a *do* loop of the following form. (The MAPLE code is explained in the glossary just before the lab)

```
>with(plots);
>for a from  1 by 2 to 7 do
>implicitplot(x^2/a^2  + y^2=1,x=-8..8,y=-8..8,
 scaling=constrained,axes=normal,grid=[100,100])
>od;
```

You may also use the following commands.

```
>with(plots);
```

```
>implicitplot({seq(x^2/(2*j-1)^2 + y^2= 1, j=1..4)},
 x=-8..8,y=-8..8,scaling=constrained,axes=normal,
 grid=[100,100]);
```

These commands run faster, but they put all the plots in the same window, so you should be sure you know which values of a go with which plots. Remember that $a = 2*j-1$. **Don't print the plots unless you need to refer to them!**

Describe the changes you see in the graph as the coefficient a increases. Say what the length of the semi-major axis is for each value of a. What is the length of the semi-minor axis?

(b) Plot $x^2 + y^2/b^2 = 1$ for $b = 1, 3, 5, 7$.

Describe the changes you see in the graph as the coefficient b increases. What is the length of the semi-major axis for each value of b? What is the length of the semi-minor axis?

Question 2. What is an equation of an ellipse whose semi-major axis lies on the x-axis, such that the semi-major axis is four times as big as the semi-minor axis?

Attach a print out of the graph of your ellipse. (Use the *implicitplot* command.)

Question 3. Plot $x^2/9 + y^2/4 = c$ for $c = -2, 0, 2, 4, 6$.

Notice that the curves you just plotted are levels $-2, 0, 2, 4, 6$ of the function $f(x, y) = x^2/9 + y^2/4$. As c increases, describe the changes you see in the plots.

Based on these plots, describe how the level curves of f change as c goes from $-\infty$ to ∞.

Hyperbolas

Question 4. Don't print the plots unless you need to refer to them!

(a) Plot $x^2/a^2 - y^2 = 1$ for $a = 1, 3, 5, 7$.

Describe the changes you see in the graph as the coefficient a increases. (Make sure you mention how the asymptotes and vertices change. If you forget what the asymptotes are or how to find their equations, look back at the bottom of page 5.)

(b) Plot $x^2 - y^2/b^2 = 1$ for $b = 1, 3, 5, 7$.

Describe the changes you see in the graph as the coefficient b increases. (Make sure you mention how the asymptotes and vertices change.)

Question 5. Find the equation of a hyperbola whose asymptotes have slope 1 and -1, and whose vertices are located at $(-6, 0)$, $(6, 0)$.

Attach a printout of the graph of your hyperbola.

Question 6. Plot $x^2 - y^2 = c$ for $c = -16, -4, 0, 4, 16$.

Describe the changes you see in the plots as the coefficient c increases. Notice that the curves you just plotted are levels $c = -16, -4, 0, 4, 16$ of the function $f(x, y) = x^2 - y^2$. Based on these plots, describe how the level curves of f change as c goes from $-\infty$ to ∞.

Recognizing Quadric Surfaces

If we have a quadratic equation of three variables, like $2x^2 + x - z - xy + 2yz = 1$ it's often hard to see what kind of surface it defines by looking at the equation. (If you go on in linear algebra you will learn a way to tell which kind of surface you have from the equation in that subject.) However, with a little practice we can use a plotting program like **Maple** to help us with the identification.

Use **Maple** and `implicitplot3d` to identify the surfaces given by the equations below. In each case attach a printout of the plot, and give the name of the surface (for example, "Hyperboloid of 1 sheet"). At least for starters, include the following commands `x=-3..3, y=-3..3, z=-3..3`, `axes=boxed, grid=[12,12,12]`. Once you get a plot, you may want to change some or all of these. Remember that you can also rotate the plot to get a better viewpoint: click on it once so that a frame forms around it (this will also give various plotting options on the menu bar at the top of the screen); now dragging the plot in various directions will cause it to rotate. Do this slowly and carefully; it takes a while to get used to how it works.

7a) $2xy - z^2 + yz - x = 1$

7b) $2x^2 + x - z - xy + 2yz = 1$

7c) $xz + 3yz = 1$

7d) $2x^2 + 3y^2 + 4z^2 - 3xy + yz = 8$

7e) $xy - 3yz + x = 1$

Finally, you can probably fit at least two of these plots on a page: this will save time and paper!