NORTHEASTERN UNIVERSITY

Department of Mathematics Fall '03

Gradient Fields Lab

Goals

By the end of this lab you should:

1.) Be able to recognise critical points of a function $f(x, y)$ from the plot of the gradient field.

2.) Be able to draw the paths taken by particles that are moving tangent to the gradient field, or that are moving in the direction opposite to the gradient field.

3.) Understand one of the ways that gradient vector fields come up in physics and engineering.

4.) Be able to find stable equilibrium points of conservative force fields based on a plot of the gradient of the potential energy.

Introduction

In class, you have learned that the gradient vector of a function at a point points in the direction of greatest increase of the function. Remember that the gradient is the vector of partial derivatives. For most functions f in this course, the gradient vector will be defined at every point of the domain of f. The set of all of these vectors is called the **gradient field** of f. Given $f(x, y)$, you can visualize the gradient field of f as a bunch of arrows, one arrow at each point of the plane. The concept of the gradient field is very important in science and engineering. Every conservative force field, such as the force due to gravity, or electric charge is a gradient field. Gradient fields are also used to describe fluid flow in mechanical and civil engineering.

The points where the gradient of f is the zero vector are called **critical points**. The pattern of the gradient field around these points can be quite complicated. In this lab, you will get a feeling for the typical patterns of the gradient field around critical points, and some practice thinking about the important properties of the gradient vector. At the end of the lab, you will apply what you have learned to a problem in electrostatics.

Level Curves and Gradients Close to the Critical Points:

There are three kinds of critical points that occur most often: maxes, mins and saddles. Here are three functions, each with a critical point at the origin.

$$F_{Min}(x, y) = x^2 + y^2$$

$$F_{Max}(x, y) = -x^2 - y^2$$

$$F_{Sad}(x, y) = x^2 - y^2$$

The first function has its *minimum value* at the origin, the second function has its *maximum value* at the origin, while the third has a *saddle point* there. (At a saddle point, you can find two perpendicular lines such that along one line the function has a local max, and on the other line it has a local min. F_{Sad} has a local min along the x axis and a local max along the y axis. The graph of a function at such a point looks like a saddle, hence the name.)

Start MAPLE and use the following commands to show the levels and gradient field at the same time for F_{Min}:

```
>with(plots);
>FMin:=(x,y)->x^2+y^2;
>A:=implicitplot({seq(FMin(x,y)=.2*j, j=1..5)},
 x=-1..1,y=-1..1,axes=boxed,scaling=constrained):
>B:=gradplot(FMin(x,y),x=-1..1,y=-1..1,arrows=SLIM):
>display({A,B},axes=boxed,scaling=constrained);
```

The third command produces the numerical data necessary to draw the graph of the equation $F_{Min}(x, y) = .2 * j$ as j goes from 1 to 5. The fourth command produces the numerical data necessary to draw a plot of the gradient field of F_{Min}. The fifth command converts the numerical data into a single sketch. By putting a colon at the end of the second and third commands, you prevent your worksheet from being cluttered with a printout of the numerical data. (**Remember that MAPLE is case sensitive**; you will **not** get any pictures if you use "Display" instead of "display".)

Question 1: Describe the important features of your plot in two sentences. On your plot, trace the path of your trajectory if you start at (.1,0) and the trajectory is always tangent to the gradient field.

Question 2: By hand, plot the gradient field and level curves of F_{Max}. What do the plots for F_{Min} and F_{Max} have in common? How are they different? On your plot trace the trajectory of a point moving tangent to the gradient field starting at (1,0). How could you tell just by looking at the two plots that one function had a maximum value at (0,0) and the other a minimum?

Question 3: Now use MAPLE to plot the levels and gradient field of F_{Sad} as you did with F_{Min}. You will have to enter F_{Sad}, and you will have to change the values of the levels plotted to get a good picture. We suggest you use $F_{Sad}(x, y) = -.8 + 0.16 * j, j = 1..10$. What do most of the levels look like? On your plot, draw the trajectories tangent to the gradient field starting from $(.1, 0)$, $(-.1, 0)$, $(0, .9)$, $(0, -.9)$, $(.1, .9)$, $(-.1, .9)$, $(.1, -.9)$, $(-.1, -.9)$. On your plot, indicate which of your trajectories look like hyperbolas and which of your trajectories end at the origin. Pick a trajectory that ends at the origin, and explain why this happens on your plot.

Potential Energy and Stable Equilibrium Points

We are now going to apply what we've learned to a problem in electrostatics. Suppose we have a square plate with sides two units long, with fixed unit positive charges located at the four corners. We also have a unit charge which is allowed to move on the plate. Are there any points inside the square of stable equilibrium–that is, are there points, (x, y), where no force acts on the charge, and if the charge is sitting at (x, y) and we disturb the charge a little the charge moves back to (x, y)? Let's find out.

Assume the square lies in the xy-plane and that the corners of the square are located at $(-1, -1)$, $(-1, 1)$, $(1, -1)$, $(1, 1)$. We know from physics that if we have a charge of strength Q then the electrical potential P of a unit charge is

$$P = Q/r$$

where r is the distance between the charges.

In our problem, we have a unit positive charge at (x, y). So the electrical potential due to our four unit charges in the corners is just the sum of the potential due to each charge. The potential due to the charge at $(1, 1)$ is just $1/\sqrt{(x-1)^2 + (y-1)^2)}$, so when we add up the potentials coming from all four charges, we get:

$$\begin{aligned} P(x, y) = & \frac{1}{\sqrt{(x-1)^2 + (y-1)^2)}} + \frac{1}{\sqrt{(x-1)^2 + (y+1)^2)}} \\ & + \frac{1}{\sqrt{(x+1)^2 + (y-1)^2)}} + \frac{1}{\sqrt{(x+1)^2 + (y+1)^2)}}. \end{aligned}$$

Warm-up question: What would the potential be if the charge at $(1, 1)$ was 2 units instead of 1?

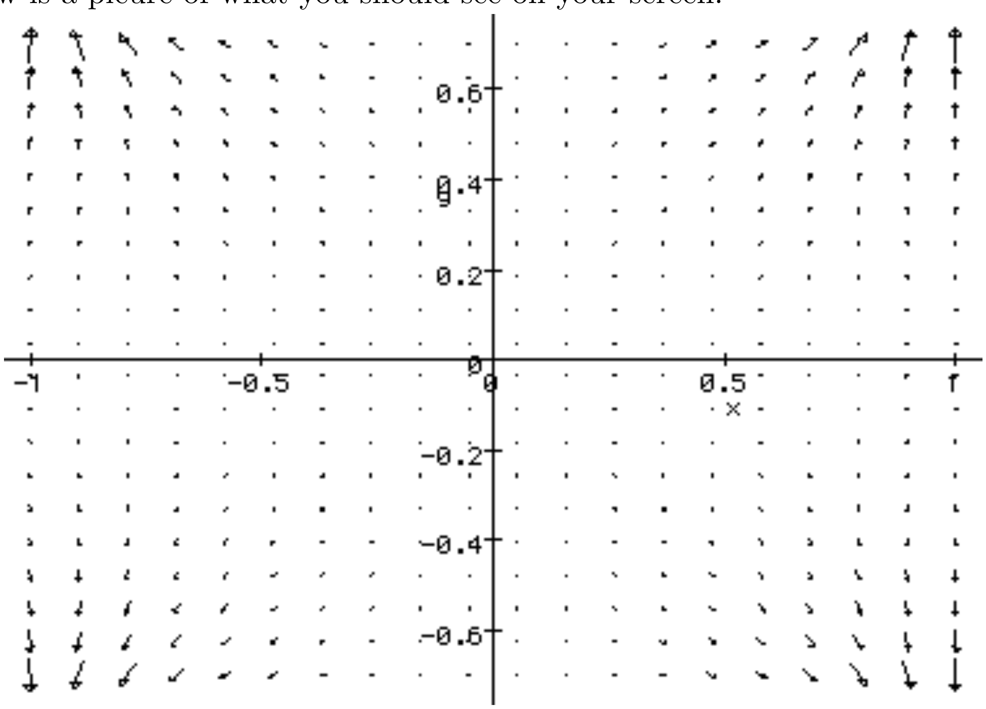
In physical situations like the one above, there is usually a **dissipative** force as well. This force acts like the force of friction for a ball running down a hill. The dissipative force drains energy from the system. As a result of the force due to the electric field and the dissipative forces acting on the particle, if we release a unit charge from rest, it will try to move so as to minimize its potential energy, just as a particle in a gravitational field, when released, tries to minimize its potential energy. If the particle is sitting at a point where the potential energy has a local minimum, then if we move it a little it willl move back to the local min. This means that the local equilibrium points will be where the potential energy has a local minimum.

Note that because the particle is trying to minimize it's potential energy, the force acting on the particle is always **opposite** the gradient of the potential energy.

This discussion suggests that to find the equilibrium points, we plot the gradient field of P, and look for local mins. Since the potential energy is infinite on the corners, there cannot be a local min there, so we will look at the gradient field on a smaller rectangle which cuts off the corners. Please enter the following commands.

```
> P:=(x,y)->((x-1)^2+(y-1)^2)^(-1/2)+((x+1)^2+(y-1)^2)^(-1/2)+
((x+1)^2+(y+1)^2)^(-1/2)+((x-1)^2+(y+1)^2)^(-1/2);
>gradplot(P(x,y),x=-1..1,y=-.7..0.7,arrows=SLIM);
```

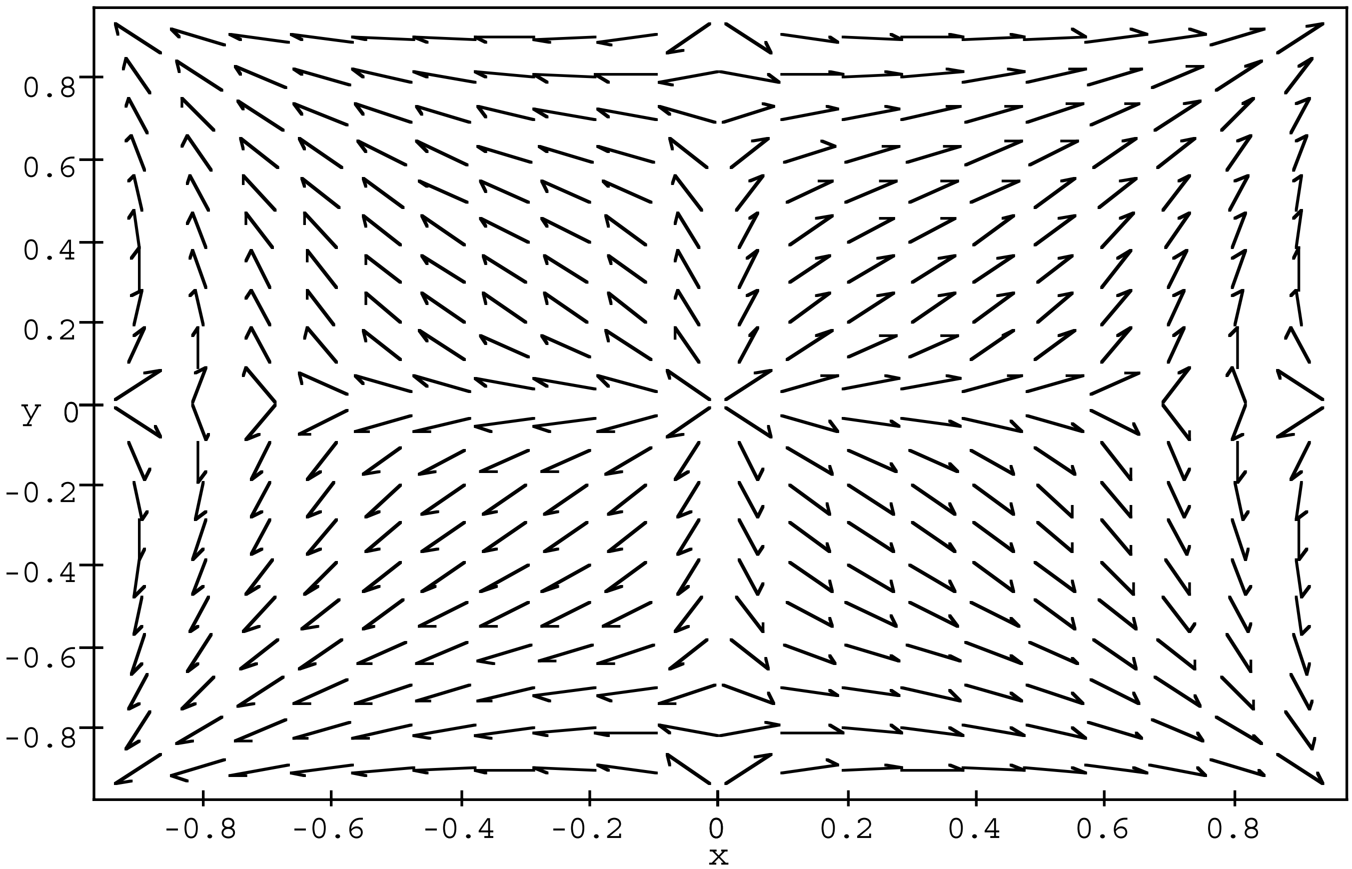
Below is a picure of what you should see on your screen:



The gradient plot you get isn't terribly clear. MAPLE has a problem here, because the magnitude of the gradient vectors varies so much. When MAPLE tries to draw the vectors which are small in magnitude in the same scale as the vectors that are large in magnitude, the small vectors look like points. One way to deal with this problem, is to change the magnitude of the gradient vectors so that they are all unit vectors. We call the new vector field we get this way the **unit gradient field.** The next set of commands will plot the unit gradient field, by dividing each gradient vector of P by its magnitude, and then plotting this vectorfield.

```
>p:=diff(P(x,y),x);
>q:=diff(P(x,y),y);
>fieldplot([p/(p^2+q^2)^(1/2),q/(p^2+q^2)^(1/2)],x=-0.9..0.9,
y=-0.9..0.9, axes=boxed);
```

Here is what your screen will show when these commands are correctly executed.



There's no problem seeing the vectors now; we just have to understand what we are looking at. If P has a minimum value at (x, y), then the gradient vectors will all point away from (x, y). It does look like the gradient vectors are pointing away from $(0, 0)$.

Let's take a closer look. Please enter:

```
>fieldplot([p/(p^2+q^2)^(1/2),q/(p^2+q^2)^(1/2)],x=-0.1..0.1,
y=-0.1..0.1, axes=boxed);
```

Question 4: Based on this plot, what kind of a critical point of P is $(0, 0)$? Explain what would happen if we released a unit charge at $(.05, .05)$, and the charge always moved in the direction of the force acting on it. (Remember the force is **opposite** the gradient of the electrical potential).

Question 5: The potential energy also has 4 saddle points on the original square; circle these four points on the plot of the unit gradient field of P. (If you are unsure which points these are, you can have MAPLE zoom in on points that look suspicious, the same way we looked more closely at $(0,0)$.)

Question 6: Suppose we increase the size of the charges on the lower left and upper right corners to 4 units. To two decimal places, at which points does the potential energy have a minimum now? Hint: there are two local mins; they lie on the line $y = -x$, and they are both the same distance from the origin. (To answer this, change the potential function to reflect the change in charge, and again plot the unit gradient field on the square $x = -0.9..0.9, y = -0.9..0.9$. Now look more closely at one of the points at which P seems to have a local minimum value. Create a plot which shows where this point is to within .01, and which also shows that it is a local minimum of P.)