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Problem 1. Suppose V is to be found from the formula $V = \frac{T}{P+T}$, where T and P are found to be 9 and 1 with maximal possible error $|dT| = 0.5$ and $dP = 0.1$. Estimate the maximum possible error in the computed value of V.

Solution.

$$V_T(T,P) = 1/(P+T) - T/(P+T)^2 \implies V_T(9,1) = 0.01.$$
$$V_P(T,P) = -T/(P+T)^2 \implies V_P(9,1) = -0.09.$$
$$dV = V_T(9,1)dT + V_P(9,1)dP \le |V_T(9,1)dT| + |V_P(9,1)dP| \le 0.01 \cdot 0.5 + 0.09 \cdot 0.1 = 0.014.$$

Answer: The maximal possible error is 0.014.

Problem 3. If $r = 10.0$cm and $h = 8.0$cm to the nearest millimeter, what should we expect the maximum percentage error in calculating $V = \pi r^2 h$?

Solution.

$$V_r(r,h) = 2\pi rh, \quad V_h(r,h) = \pi r^2.$$
$$dV = V_r dr + V_h dh \implies \frac{dV}{V} = \frac{V_r}{V}dr + \frac{V_h}{P}dP = \frac{2dr}{r} + \frac{dh}{h} \le \frac{2 \cdot 0.1}{10} + \frac{0.1}{8} \approx 0.03.$$
$$\frac{dV}{V} \cdot 100\% \le 3\%.$$

Answer: The maximal percentage error is 3%.

Problem 5. Find and classify the critical points of the function

(a) $f(x,y) = xy(x+y-1)$;

Solution.

$$f_x = y(x+y-1) + xy = 0 \implies y = 0 \text{ or } x+y-1+x = 2x+y-1 = 0.$$
$$f_y = x(x+y-1) + xy = 0 \implies x = 0 \text{ or } x+y-1+y = x+2y-1 = 0.$$

If $y = 0$ then the second equation implies $x = 0$. If $x = 0$ the first equation implies $y = 0$. Thus we have one critical point $(0,0)$. The other critical points must satisfy

$$2x+y-1 = 0 \quad \text{and} \quad x+2y-1 = 0.$$

Solving this 2 equations we obtain $x = 1/3, \ y = 1/3$.

Thus there are 2 critical points $(0,0)$ and $(1/3,1/3)$.

$$f_{xx} = 2y, \quad f_{yy} = 2x, \quad f_{xy} = f_{yx} = 2x+2y-1.$$

(0,0) $f_{xx}(0,0) = 0, \ f_{yy}(0,0) = 0, \ f_{xy}(0,0) = -1$. Thus

$$D(0,0) = f_{xx}(0,0)f_{yy}(0,0) - f_{xy}(0,0)^2 = -1 < 0 \implies (0,0) \text{ is a saddle point.}$$

(1/3,1/3) $f_{xx}(1/3,1/3) = 2/3, \ f_{yy}(1/3,1/3) = 2/3, \ f_{xy}(1/3,1/3) = 1/3$. Thus

$D(0,0) = f_{xx}(0,0)f_{yy}(0,0) - f_{xy}(0,0)^2 = 1/3 > 0; \ f_{xx}(1/3,1/3) = 2/3 > 0 \implies (1/3,1/3)$ is a minimum.

Answer: The function has a saddle point at (0,0) and a minimum at (1/3,1/3).