Supplementary problems

Drawing graphs: Some problems

1) Draw the graphs of the following functions. Remember to sketch the part of the graph over the x and y axes, choose a “spine”, then draw some “ribs”. Drawing a final sketch after doing a rough sketch will give you a better sketch.

A) $f(x,y) = -3 + x^2 + y^2$

B) $f(x,y) = 1 - x^2 - y^2$

C) $f(x,y) = 2 - x^2 + y^2$

D) $f(x,y) = -3 + x^2 - y^2$

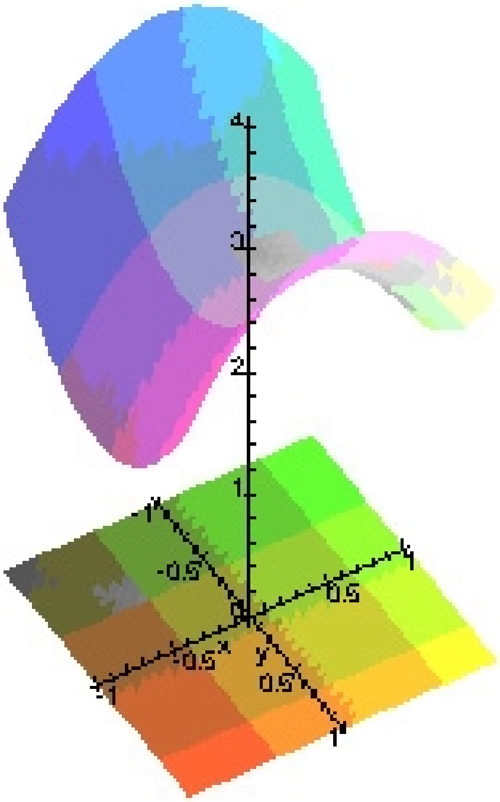
2) Match the function with its graph.

A) $f(x,y) = 1 + x^2 + y^2$, $-1 \leq x \leq 1, -1 \leq y \leq 1$

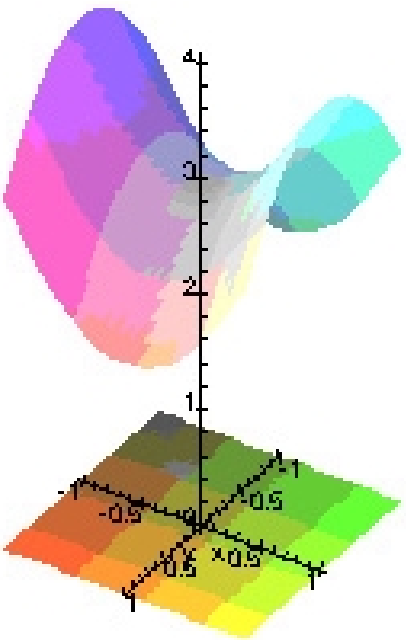
B) $f(x,y) = 3 - x^2 - y^2$, $0 \leq x \leq 1, -1 \leq y \leq 1$

C) $f(x,y) = 3 - x^2 + y^2$, $-1 \leq x \leq 1, -1 \leq y \leq 1$

D) $f(x,y) = 3 + x^2 - y^2$, $-1 \leq x \leq 1, -1 \leq y \leq 1$



I



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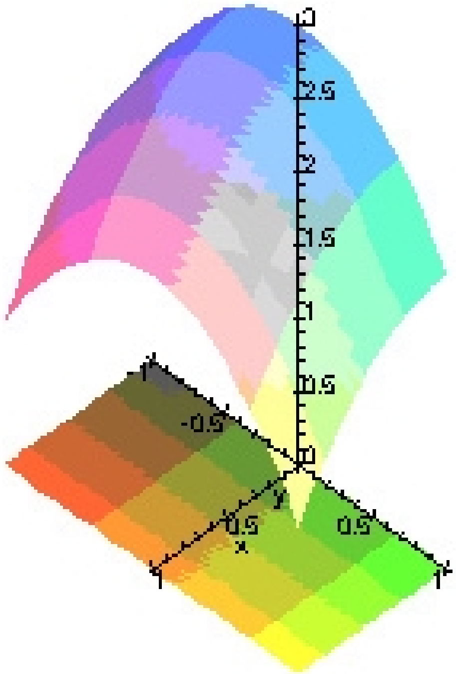

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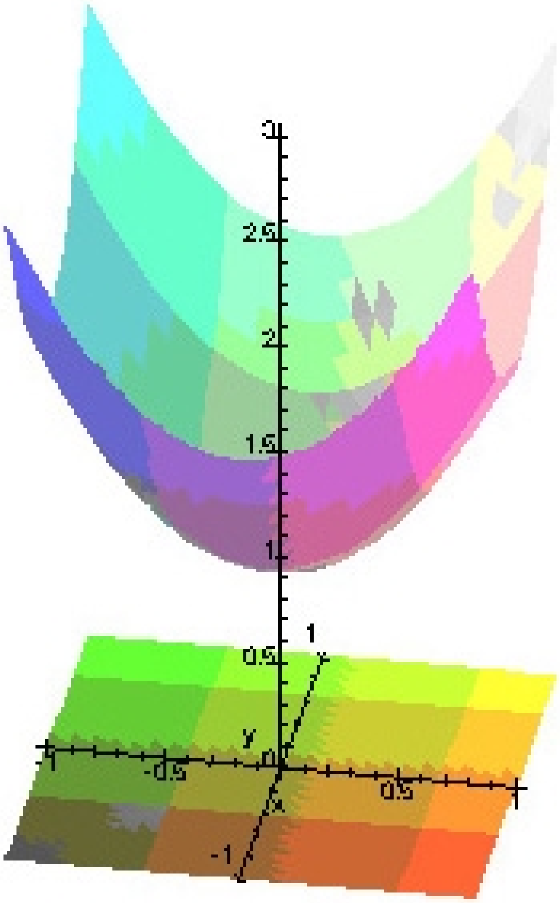


IV

3) In this problem we are going to give you some practice in drawing the graphs of functions whose domain is a rectangle. These exercises will teach you how to break these graphs into parts, then to connect the parts to get a good sketch. To make it easy for you, the answer will be one of the plots in the last problem. Here's how to do it. Decide on what the graph looks like, without worrying about the domain. Then, draw the part of the graph over each of the sides of the rectangle. Now, add additional curves based on the kind of graph it is. Compare your sketch to the MAPLE plot to see how you can improve your sketch.

A) Sketch the graph of $f(x,y) = 1 + x^2 + y^2$, $-1 \leq x \leq 1, -1 \leq y \leq 1$

B) Sketch the graph of $f(x,y) = 3 - x^2 - y^2$, $0 \leq x \leq 1, -1 \leq y \leq 1$

C) Sketch the graph of $f(x,y) = 3 - x^2 + y^2$, $-1 \leq x \leq 1, -1 \leq y \leq 1$

D) Sketch the graph of $f(x,y) = 3 + x^2 - y^2$, $-1 \leq x \leq 1, -1 \leq y \leq 1$

4) For each equation, first sketch the points in the xy plane that satisfy the equation, then sketch the points in space that satisfy the equation.

A) $x^2 + y^2 = 25$

B) $y = 100 - x^2$

C) $y = 2x$

D) $x + y = 1$

5) In problem 4 you sketched the cylinder over a curve in the xy-plane. As this problem shows, the curve dosn't have to lie in the xy-plane.

A) Sketch the graphs of $f(x, y) = 3 - x^2 - y^2$ and $g = 1 + x^2 + y^2$, being careful to label the curve of intersection.

B) Sketch the cylinder over the curve of intersection.

C) What is the equation of this cylinder?

We will use this problem later to help us to integrate over the solid bounded by the two graphs.

6) For each function sketch the graph over the given domain. Since these are linear functions, you can draw the graphs by plotting the points over the corners of the polygons, then connecting the points with line segments.

A) $z = 20 - x$, $0 \leq x \leq 10, -20 + x \leq y \leq 20 - x$

B) $z = 20 - x - 2y$, $0 \leq x, 0 \leq y$, $x + y \leq 10$

7) (Challenge problem) Read over problem #38 on page 755 and draw the part in the first octant of the solid described there. If you understand how to draw this part of the figure doing #38 will be easy. (Hint: Try to understand the part of the solid in this octant where $x = y$.)

Projection problems

1) The points $(0, 0)$, $(10, 0)$, $(10, 4)$ form a triangle. This triangle is the crossection of a triangular ramp–all measurements in meters. Resting on the ramp there is a 50 kilogram load.

a) draw a picture showing the crossection of the ramp, the force due to the load, the force along the ramp due to the load and the force perpendicular to the ramp due to the load.

b) Write the force due to the load in terms of $\vec{i}$ and $\vec{j}$.

c) Find a vector in the direction of the ramp (write this vector in terms of $\vec{i}$ and $\vec{j}$).

d) Find the force along the ramp (write this vector in terms of $\vec{i}$ and $\vec{j}$).

e) Find the force perpendicular to the ramp (write this vector in terms of $\vec{i}$ and $\vec{j}$).

2) Write down the formula for the projection of a vector $\vec{v}$ onto a vector $\vec{w}$. Explain why the formula is true (you can use a diagram as part of your answer.)

3) An electrical current is moving in a straight wire through a constant magnetic field. A segment of wire L passes through the points $(4, 0, 2)$ and $(7, -4, 3)$, and the magnetic field $\vec{M}$ is given by $\vec{M} = 4\vec{j} + 3\vec{k}$ milligauss.

a) Find a vector in the direction of the wire (Write it in terms of $\vec{i}$, $\vec{j}$ and $\vec{k}$). b) Find the component of the magnetic field in the direction of the wire. c) The magnitude of the force $\vec{F}$ (in suitable units) acting on the segment L is given by

$$\|\vec{F}\| = \|\vec{M}_{\perp}\| \cdot I \cdot l$$

, where $\vec{M}_\perp$ is the component of the field $\vec{M}$ perpendicular to the wire, I is the current and l the length of segment L. If $I = .5$ amps, what is the magnitude of the force exerted on segment L by the magnetic field?

Linearization Problems

1) If the radius of a cylinder is increased by 2%, and the height of a cylinder is decreased by 3%, use the differential to approximate the percent change in volume. (Remember that the percent change in height is $\frac{dh}{h}$ expressed as a percentage).

2) The following table gives the temperature readings T at various positions on a plate in degrees Celsius, x and y measured in centimeters.

	$y = 3$	$y = 3.5$	$y = 4$	$y = 4.5$
$x = .5$	35	34.5	33	31
$x = 1$	36.5	35	33.5	33
$x = 1.5$	36	35.5	34	33.5

a) Use the table to find approximate values for $\frac{\partial T}{\partial x}(1, 4)$, $\frac{\partial T}{\partial y}(1, 4)$.
b) Using the table and your answers to a) give the local linearization for T at $(1, 4)$.
c) Use the local linearization to approximate the change in T if x increases by .1cm and y decreases by .2cm .

3) The density ρ of carbon dioxide gas depends on its temperature T (in ° C) and pressure p (in atmospheres). The ideal gas model for this gas gives what is called the state equation:

$$\rho(p, T) = \frac{0.5363p}{T + 273.15}$$

a) Compute the differential $d\rho$.
b) Use the differential to estimate the change in density that the gas would undergo if its temperature changed from 45° to 50° and its pressure changed from 4 atmospheres to 3.9.

4) You want to measure the height of a tower. You fix a starting point, and measure the distance d from your starting point to the tower as 100ft. You measure the angle θ between the ground and a line running from your starting point to the top of the tower as $\pi/4$ radians. Given the accuracy of your tools, you figure that the error Δd in measuring d is .1 ft, and the error $\Delta\theta$ in measuring θ is $\pi/400$. Find h, the height of the tower, and use the formula for the differential of h to estimate the error in your estimate of h.

(Helpful formulas $\dfrac{d\tan\theta}{d\theta} = \sec^2\theta$, $\dfrac{d\cot\theta}{d\theta} = -\csc^2\theta$.)

5) A chemical reaction combines substances BrO_3 and Br and H. If BrO_3 has concentra-

tion x mol./l and Br has concentration y mol./l, and H has concentration z mol./l, the rate R in mol./l/secs. of change in the concentration of BrO_3 is given by

$$R = 8xyz^2.$$

Use the differential (or a linear approximation) to estimate the percent change in the reaction rate when x increases by 3%, y increases by 2% and z decreases by 3%.

6) Assume that f is differentiable at (a, b) (see pages 652 and 689 in the text). Define the linearization of f. What is its significance? Explain what the conection between the linearization of f and the differential of f is. (Hint: the differential has something to do with the change in the linearization.) Use this connection to explain why we can use the differential of f to approximate the change in $f(x, y)$ at (a, b). Define the percent change in f and explain how to approximate this using the differential.

Newton's method in 2 variables

When you studied the derivative for the first time, one of the applications you studied was Newton's method for finding approximate solutions to an equation like:

$$x^2 + \sin(x) = 7x$$

In this course we have stressed the importance of being able to linearize a function of two variables using its first Taylor polynomial. In this section we will give another reason why linearization is important. We will use the notion of linearization to extend Newton's method to systems of two equations in two unknowns. If you understand what is going on here you should be able to develop a method that works for any system of equations with the same number of equations as variables.

Here is our situation. We have two non-linear equations in two unknowns:

$$f(x, y) = c_1$$

$$g(x, y) = c_2$$

and an initial guess for the solution (a, b). Our strategy is to replace f by the local linearization $T(f)$, g by $T(g)$. These substitutions will change the non-linear equations to linear equations, then we solve the linear equations and repeat the process. By repetition, we hope to get a point as close to the true solution as we want.

Begin by replacing f and g, by their first Taylor polynomials $T(f)$ and $T(g)$ at (a, b). This will give us a set of equations:

$$T(f) = c_1$$

$$T(g) = c_2$$

If we write out $T(f)$ and $T(g)$, we get:

$$f(a,b) + \left(\frac{\partial f}{\partial x}(a,b)\right) \cdot (x-a) + \left(\frac{\partial f}{\partial y}(a,b)\right) \cdot (y-b) = c_1$$

$$g(a,b) + \left(\frac{\partial g}{\partial x}(a,b)\right) \cdot (x-a) + \left(\frac{\partial g}{\partial y}(a,b)\right) \cdot (y-b) = c_2$$

Because the Taylor polynomials of f and g are close to f and g, the solutions to the new equations should be close to the solutions of the old equations. Now you can re-write these equations as

$$\left(\frac{\partial f}{\partial x}(a,b)\right) \cdot \Delta x + \left(\frac{\partial f}{\partial y}(a,b)\right) \cdot \Delta y = c_1 - f(a,b)$$

$$\left(\frac{\partial g}{\partial x}(a,b)\right) \cdot \Delta x + \left(\frac{\partial g}{\partial y}(a,b)\right) \cdot \Delta y = c_2 - g(a,b)$$

where $\Delta x = (x-a)$ and $\Delta y = (y-b)$. These equations are linear in Δx and Δy and we can solve them by hand, or if there are more variables, by machine, then repeat. It is only because the first Taylor polynomial is a linear function that we get the linear equations we need to make the method work. Here's an example to make the algebra more concrete.

Example 1: Consider the system of non-linear equations

$$f(x,y) = x^2 + y^2 = 2$$

$$g(x,y) = y - x^2 = 0$$

with initial guess $(0.9, 0.9)$. By inspection, you can see that $x = 1$, $y = 1$ is a solution to the system. Let's see how close we can get applying our algorithm once. We have that $\frac{\partial f}{\partial x}(0.9, 0.9) = 1.8$, $\frac{\partial f}{\partial y}(0.9, 0.9) = 1.8$, $\frac{\partial g}{\partial x}(0.9, 0.9) = -1.8$, $\frac{\partial g}{\partial y}(0.9, 0.9) = 1$, $f(0.9, 0.9) = 1.62$ and $g(0.9, 0.9) = .09$ so we get

$$1.8\Delta x + 1.8\Delta y = 2 - 1.62 = .38$$

$$-1.8\Delta x + \Delta y = 0 - .09 = -.09$$

Now we need to solve this set of equations for Δx and Δy. We need to eliminate one of the variables; we can do this by adding the two equations together. This gives

$$2.8\Delta y = .29.$$

Solving for Δx in the first equation gives

$$\Delta x = (.38/1.8) - \Delta y.$$

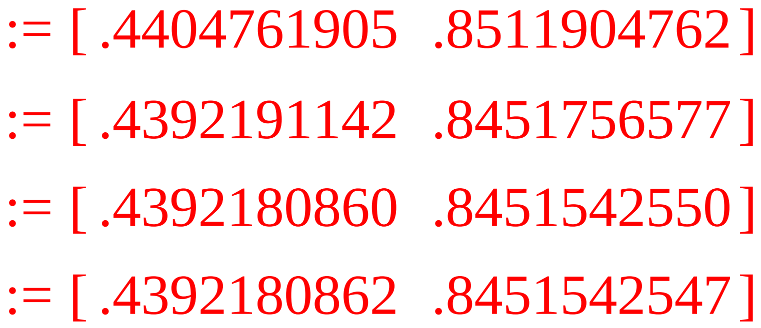
So we get $\Delta y = .1035714286$, $\Delta x = .1075396825$. Since $\Delta x = x - .9$, we get that our new x value is $.1075396825 + .9 = 1.007539683$, while the new y value is 1.003571429.

Example 2: Consider the system

$$4x^2 + 4y^2 + xy = 4$$

$$3y^2 - xy - 4x^2 = 1.$$

We used MAPLE's implicitplot command to plot the curves $4x^2 + 4y^2 + xy = 4$ and $3y^2 - xy - 4x^2 = 1$. This plot showed that the equations had 4 solutions as these curves crossed in four places. Clicking on one of the points of intersection gave a starting value of $(.5, .75)$. Doing the algorithm 4 times with MAPLE gave the following values for the solution close to $(.5, .75)$:



```
:= [ .4404761905  .8511904762 ]
:= [ .4392191142  .8451756577 ]
:= [ .4392180860  .8451542550 ]
:= [ .4392180862  .8451542547 ]
```

Given that we know that there is a root close by, and the last two answers agree to eight places, it seems reasonable to trust the final answer to 8 places. In the homework, you will have a chance to work through two examples with MAPLE.

If we are applying Newton's method to a system of equations of the form

$$f(x, y) = 0$$

$$g(x, y) = 0$$

with inital guess (a, b), there is a geometric way to think of the algorithim. The solution to the original system of equations is where level 0 of f intersects level 0 of g. This is just where the two graphs $z = f(x, y)$, $z = g(x, y)$ and the x,y plane intersect. When we apply Newton's method, we replace f and g by their first Taylor polynomials at (a, b). When we solve this new system of equations, we get the point where the x,y plane, the tangent plane to the graph of $z = f(x, y)$ at $(a, b, f(a, b))$, and the tangent plane to the graph of $z = g(x, y)$ at $(a, b, g(a, b)$ all intersect. This is because the graph of the first Taylor polynomial of f at (a, b) is exactly the tangent plane of f at $(a, b, f(a, b))$. Since the tangent planes are close to the graphs, the point turned up by Newton's method is close to the actual root.

Problem 1: Given the system of equations

$$4x^2 + xy = 5$$

$$x^2 + y^2 = 2$$

and a starting value of $(.9, .9)$, use Newton's method once to improve your estimate of the root close to $(.9, .9)$

Problem 2: Given the system of equations

$$4x^2 - xy + y^2 = 4$$

$$x^2 - 2y^2 = -1$$

and a starting value of $(1.1, .9)$, use Newton's method once to improve your estimate of the root close to $(1.1, .9)$.

Problem 3: In your own words describe the steps you go through in Newton's method. (Pretend you're writing directions for a not-so swift friend who wants to use the method to solve a problem.)

Newton's Method Computer Problems

As a first example, let us consider trying to solve

$$\begin{aligned} f = 3x^2 - xy &= 8 \\ g = y^2 + 5x^2 &= 12. \end{aligned}$$

You can check that $(\sqrt{2}, -\sqrt{2})$ is a solution of this system, but let's see how fast Newton's method approximates the solution if we start at $(1.4, -1.4)$. We'll apply Newton's method five times, letting MAPLE do all the computations.

Implementing Newton's Method in MAPLE

Warning: In entering the code that follows you have to be very careful not to make a typo, as MAPLE is powerful enough to carry mistakes along as new variables, resulting in pages of gibberish.

Start MAPLE, and enter f and g with

```
>f:=3.0*x^2-x*y;
>g:=y^2+5*x^2;
```

We need to enter the initial x and y values and the right hand side of our system of equations. Our initial guess is $(1.4, -1.4)$.

```
> righthandside_of_eq_1:=8;
> righthandside_of_eq_2:=12;
> NewX:=1.4;
```

```
> NewY:=-1.4;
```

Here righthandside_of_eq_1 and righthandside_of_eq_2 are the right hand sides of our equations, while NewX and NewY are the initial x and y values.

We next load the package that MAPLE uses to calculate the local linearization of a function (Also known as the first Taylor polynomial of the function.)

```
> readlib(mtaylor):
```

We also want MAPLE to carry 100 digits in all of its computations so we enter:

```
> Digits :=100;
```

So that we can see how accurate Newton's method is, we want to calculate $\sqrt{2}$ to a hundred places, so enter:

```
> evalf(2^(1/2));
```

Now we are ready for the main part of the program.

```
>for k from 1 to 5 do
>    Tf:=mtaylor(f,[x=NewX,y=NewY],2):
>    Tg:=mtaylor(g,[x=NewX,y=NewY],2):
>      s:=solve({Tf=righthandside_of_eq_1, Tg=righthandside_of_eq_2},
{x,y}):
>    assign(s):
>    NewX:=x;
>    NewY:=y;
>    x:='x':
>    y:='y':
> od;
```

The first line of the program sets up a loop with counter k. The second line of the program calculates the local linearization of f, using NewX and NewY for the center of the expansion. The 2 in the parenthesis tells the program to use only the constant term and the linear term in the Taylor expansion of f. The third line does the same for g. The fourth line tells MAPLE to solve the linearized version of our equations, and call the set of solutions s; s is a set of two equations of the form $\{x = a, y = b\}$. The fifth line assigns to x and y their values from s. So, if $x = a$ is an element of s, this command will give x the value a. The next commands give NewX and NewY their new values, and turn x and y into variables again.

Problem 1. In practice, when we use a numerical process to find a root, we trust those consecutive digits in an answer which do not change when we repeat the algorithm. Call these digits, *reliable digits*. For each of the five values of NewX and NewY produced by a loop in the above example, say how many reliable digits there were. (For the fifth time through you can compare with the value of $\sqrt{2}$ produced by MAPLE.)

Problem 2. Based on your answer to Problem 1, give a rough rule which predicts the number of reliable digits you have after going through a loop, in terms of the number of reliable digits you had at the begining of the loop.

Application to Electrostatic Potentials

Now we will apply Newton's method to find an equilibrium point of an electric field. We suppose we have a square plate with sides two units long, with fixed positive charges located at the four corners. We are given that the charges at $(1,1)$ and $(-1,-1)$ have magnitude 4 units, while the charge at $(-1,1)$ has magnitude 3 units, and the charge at $(1,-1)$ has magnitude 2 units. This means that the electrical potential $P(x,y)$ of a unit charge on the plate is

$$\begin{aligned} P(x,y) = &\frac{4}{\sqrt{(x-1)^2+(y-1)^2}} + \frac{2}{\sqrt{(x-1)^2+(y+1)^2}} \\ &+ \frac{3}{\sqrt{(x+1)^2+(y-1)^2}} + \frac{4}{\sqrt{(x+1)^2+(y+1)^2}}. \end{aligned}$$

Problem 3. The third computer lab discusses the background of this situation in greater detail. After doing the third lab, you will see that both partials of P are zero at an equilibrium point. For now, just use Newton's method to find the point where both partials of P are zero to eight reliable digits, using $(.27, -.27)$ for your initial guess. (**Hint:** In working on this problem, it is helpful to remember that, if f is an algebraic expression, then $diff(f,x)$ is the command that calculates the partial of f with respect to x.) Attach a printout of your MAPLE worksheet to your answer.

Gradient Problems

In working on these problems remember the basic properties of the gradient.

1) If $\vec{u}$ is the unit vector in the direction of the gradient at (a,b) then $\vec{u}$ is the direction of greatest increase of f at (a,b).
2) Not only does the gradient of f at (a,b) give us the direction of greatest increase of f at (a,b), but the rate of change of f in this direction is $\|\nabla f(a,b)\|$.
3) The gradient of f is always orthogonal to the levels of f.
4) The rate of change of f in the $\vec{u}$ direction is $\nabla f \cdot \vec{u}$.

1.) The height H of a tract of land can be represented by the function

$$H(x,y) = 100 + .05x^2 - .2y^2 - .0001x^4,$$

where the positive y-axis points north, and the units of x and y are feet.

a) What is the gradient of H for $x = 10$, $y = 20$?
b) What is the rate of change of H in the following directions: northeast, southwest, north, northwest? In which of these directions does the land slope up? In which directions does the land slope down?
c) Find the rate of change of H in the direction of the gradient.
d) What is the direction of greatest increase of of H?

2.) The temperature of a plate is given by $T(x,y) = 30/(1 + 2x^2 + 4y^2)$. What is the gradient of T at $(3,2)$? If you are at position $(3,2)$, does the temperature increase or decrease toward the southeast? What is the direction in which temperature falls the fastest? What is the greatest rate at which the temperature can increase?

3.) The electrical potential of a unit charge is $20/\sqrt{x^2+y^2}$. What is the gradient of the potential energy at $(4,3)$? What is the rate of change of the potential at $(4,3)$ in the following directions: $\vec{r} = 2\vec{i} - \vec{j}$, $\vec{q} = 3\vec{i} + 2\vec{j}$. Does the potential increase in either of these directions? What is the greatest rate at which potential can fall?

4) The following table gives the altitude readings h at various positions on a hill, all measurements in meters, positive y axis points North.

	$y = -20$	$y = 0$	$y = 20$	$y = 40$
$x = -20$	35	30	25	20
$x = 0$	36	30	24	18
$x = 20$	37	30	23	16

a) Use the table to find approximate values for the gradient of h at (0,20).
b) What is the slope of the hill at $(0,20)$?
c) What is the rate of change of h in the Northeast direction?
d) In which direction is the hill level?
e) Which is the direction in which elevation falls the fastest?

5) Suppose a block of insulation occupies the region $0 < x < 20$, $0 < y < 2$, $0 < z < 3y^3$, where the distances are in centimeters. The temperature T at the point (x,y,z) in this solid is given by

$$T(x,y,z) = x^2 + y^2 - 2z^2$$

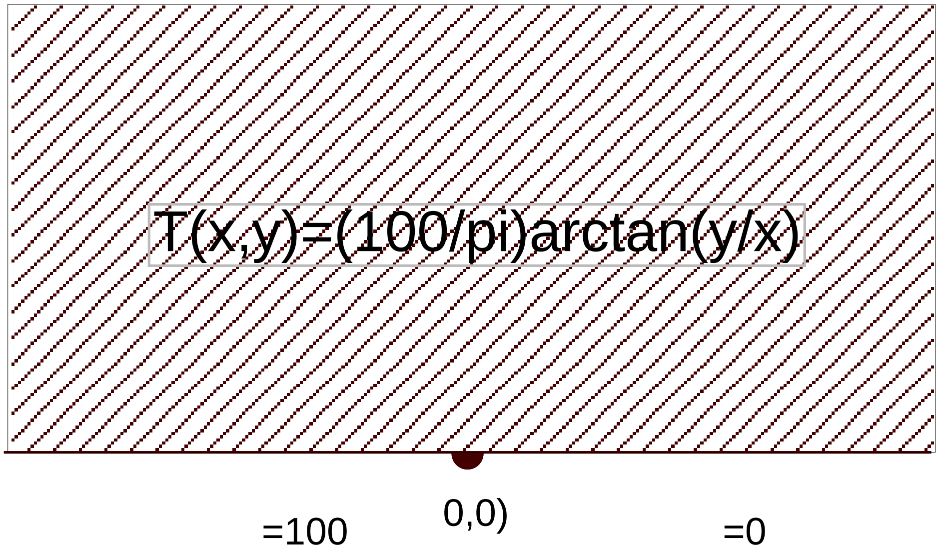
(Temperature measuered in degrees Celsius.)

a) Compute the gradient of $T(x,y,z)$ at the point $(1,1,1)$.
b) Find the rate of change of the temperature in the direction $\vec{i}+\vec{j}-\vec{k}$ at $(1,1,1)$.
c) From the point $(1,1,1)$, in what direction (unit vector) should one move so that the temperature decreases as quickly as possible?

6.) When you study heat conduction close to the edge of a sheet of metal, it's helpful to think of the edge as the x axis, and the points (x,y) with $y>0$ in the upper half plane as the rest of the metal sheet. Put a piece of insulation at the origin; keep the points on the edge with $x>0$ at $0°C$, the points on the edge with $x<0$ at $100°C$. Then the temperature T at (x,y), with x and y measured in centimeters, is given by

$$T(x,y)=\frac{100}{\pi}\arctan\frac{y}{x}$$

Here is a picture of the plate:



a) What is the gradient of the temperature at $(3,3)$? Remember that the formula for the derivative of the arctangent is $\dfrac{d\arctan(u)}{du}=\frac{1}{1+u^2}$.
b) What is the rate of change of T at the point $(3,3)$ in the direction $\vec{u}=1/2\vec{i}-\sqrt{3}/2\vec{j}$?
c) An *isotherm* is a curve on our plate along which the temperature is constant. What is the direction of the isotherm passing through $(3,3)$? (Answer with a unit vector.)
d) On our plate, heat flows along curves which are perpendicular to the isotherms; call these curves the heat flow lines. Sketch the isotherms and the heat flow lines.

7.) Suppose you are given two infinite wires which are parallel to the z-axis, passing through the points $(1,0)$ and $(-1,0)$ in the xy plane which are homogeneously and oppositely charged. The potenial energy P of a unit charge in the xy plane is given by

$$P=(1/2)\ln((x-1)^2+y^2)-(1/2)\ln((x+1)^2+y^2)$$

a) What is the gradient of the potential energy at $(3,3)$?

b) What is the rate of change of P in the direction $\vec{u} = 1/2\vec{\imath} - \sqrt{3}/2\vec{\jmath}$ at $(3,3)$?

c) An *equipotential line* is a curve along which the electrical potential is constant. What is the direction of the equipotential line passing through $(3,3)$?

8.) Looking over your notes, in your own words, give a derivation of the formula for the directional derivative. That is, I want you to show that if f is differentiable at (a,b) and $\vec{u}$ is a unit vector, then the rate of change of f in the $\vec{u}$ direction at (a,b) is $\nabla f(a,b) \cdot \vec{u}$. You should use the linearization in your proof.

9.) Use the formula for the directional derivative in #8 to show why the basic properties of the gradient listed at the beginning of this section are true. (Hint remember that $\vec{v} \cdot \vec{u} = \|\vec{v}\| \cos\theta$ if $\vec{u}$ is a unit vector.)

Additional Chain Rule Problems

1) The pressure P of an ideal gas is given by

$$P = \frac{nRT}{V}$$

where n is the number of moles of gas, T is the temperature in degrees Kelvin, and R is the gas constant 8.32 joule/mole-K°. P is measured in nt/m^2.

a) Find $\frac{\partial P}{\partial V}$, $\frac{\partial P}{\partial T}$

b) Find $\frac{\partial P}{\partial V}$ when $V = 100$ liters and $T = 300°K$, $n = 1$. What is the physical meaning of this number?

c) Suppose when $V = 100$ liters and $T = 300°K$, $n = 1$, V is decreasing at a rate of $10 liters/sec$ and T is increasing at a rate of $10 degrees/sec$. Use the chain rule to find the rate of change of P with respect to time at this moment.

2) In your first calculus course you saw that if $x(t)$ is the position of a particle moving along the x axis at time t, then $\frac{dx}{dt}$ is the velocity of a particle. If $x(t)\vec{\imath} + y(t)\vec{\jmath}$ is the displacement vector of a particle moving in the plane then its velocity vector is $\frac{dx}{dt}\vec{\imath} + \frac{dy}{dt}\vec{\jmath}$.

Suppose the particle is moving along the level curve $f(x,y) = c$. Use the Chain rule to give another proof that the velocity vector of the particle is always tangent to the level curve.

3a) A particle with a unit charge is moving in an electric field. At $t = 0$ the velocity v of the particle is $10\vec{\imath} + 3\vec{\jmath}$ and the particle is at $(3,4)$. (Velocity measured in cm/sec). If the potential energy of a unit charge due to the electric field is $P(x,y) = x^2 + y$, what is the rate of change of the potential energy with respect to time at $t = 0$? (Change in potential due to position measured in volts.)

3b) A second particle with a unit charge moves in another electric field. The velocity of the particle is $\vec{v}(t) = (3 + t^2)\vec{i} + (2 + t)\vec{j}$. At time $t = 0$, the gradient of the electrical potential energy is $4\vec{i} - 2\vec{j}$ at the position of the particle. What is the rate of change of the potential energy of the particle with respect to time at time $t = 0$?

4) The pressure P of oxygen in a bottle with a piston is given by

$$P = \frac{nRT}{V - .03n} - 1.4(\frac{n}{V})^2$$

where n is the number of moles of gas, T is the temperature in degrees Kelvin, R is the gas constant .082 L-atm/mol-K°. P is measured in atmospheres (atm).

a) Find $\frac{\partial P}{\partial V}$, $\frac{\partial P}{\partial T}$. Find $\frac{\partial P}{\partial V}$ when $V = 10$ liters and $T = 300°K$, $n = 100$. What is the physical meaning of this number?

b) Suppose when $V = 10$ liters and $T = 300°K$, $n = 100$, V is decreasing at a rate of .5 liters/sec and T is increasing at a rate of 2 degrees/sec. Use the chain rule to find the rate of change of P with respect to time at this moment.

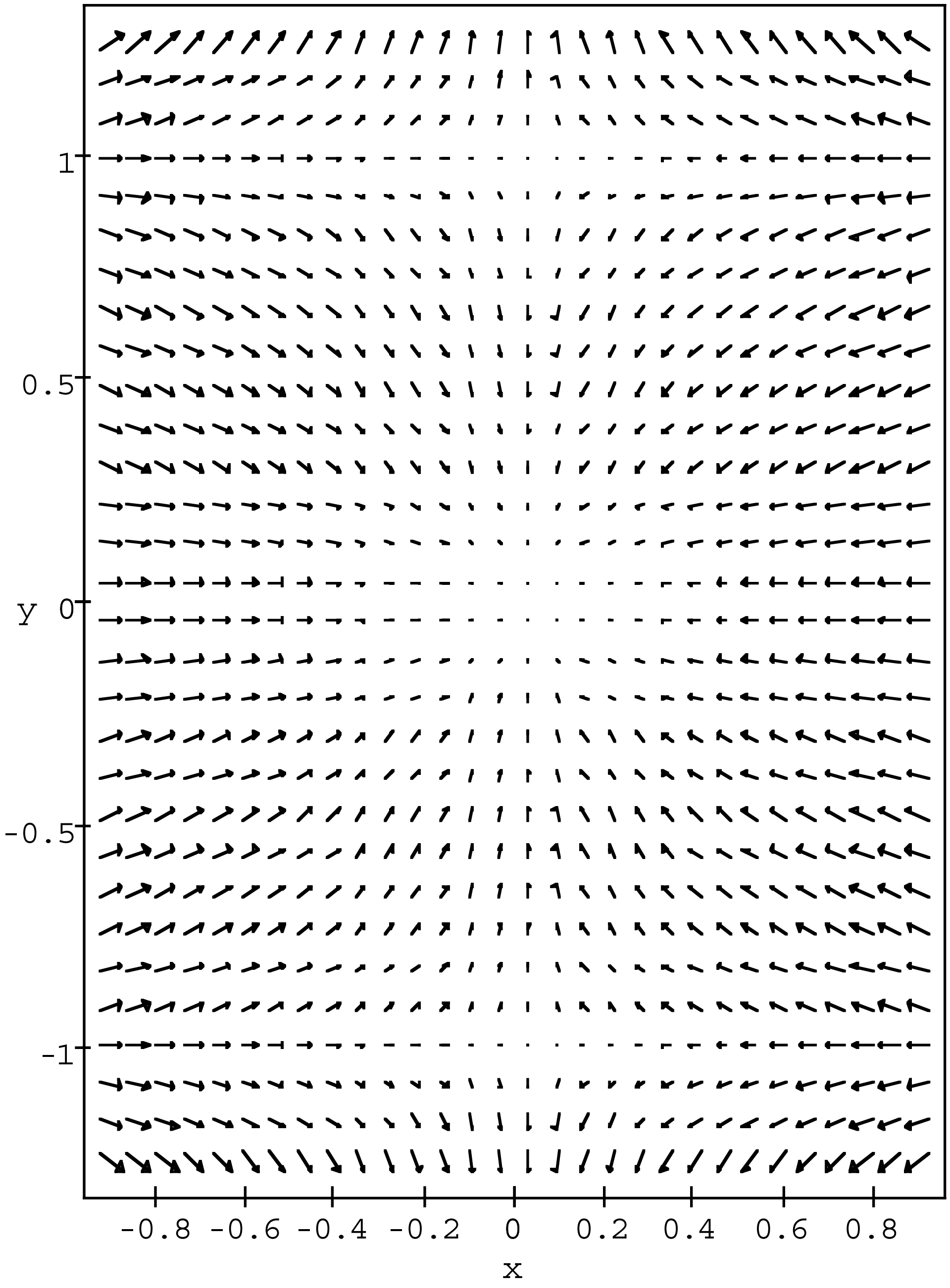
Classifying critical points using level curve plots and gradient fields

Level curve plots and gradient plots can be very helpful in finding and classifying critical points. If we are at a local minimum for a function f, then no matter which direction we turn, the function must increase, so all the gradient arrows will point away from such a point. The reverse reasoning shows that at a local maximimum, all the gradient vectors of f near the local maximum must point toward the local maximum. At a saddle, the trajectories of the gradient field will look like hyperbolas close to the saddle point.

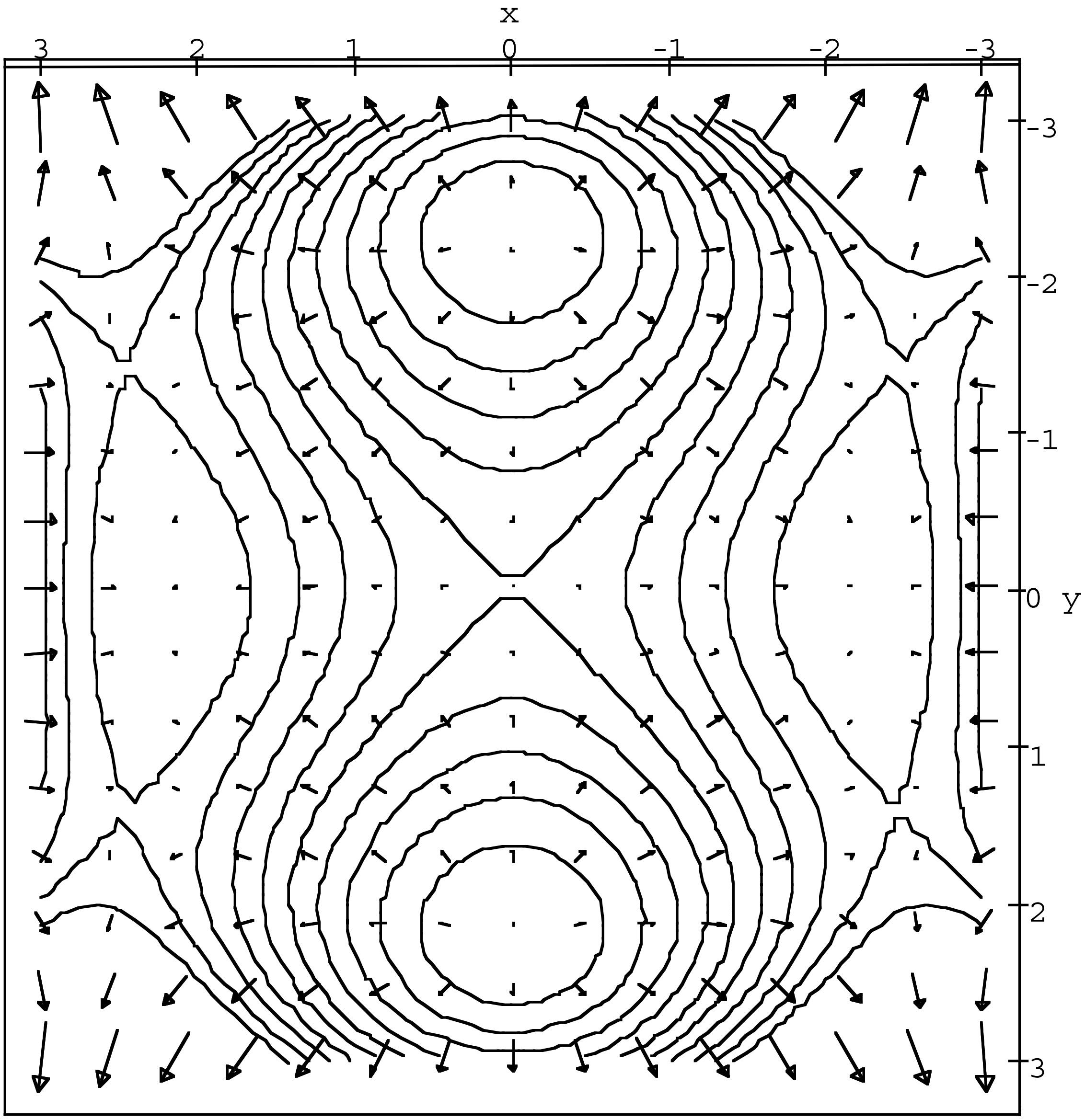
Around both a local minimum and local maximum the level curve plots will look like concentric ellipses; we need to know the value of f on the levels, or have a plot of the gradient field to distinguish a minimum from a local maximum. At a saddle point, we know that the levels also look like hyperbolas.

You have to be a little careful about relying on plots alone. If you use a large scale, it is possible for two or more critical points to be so close together that they appear to be a single critical point. With that caution, here are some exercises to give you some practice in detecting and classifying critical points by plots.

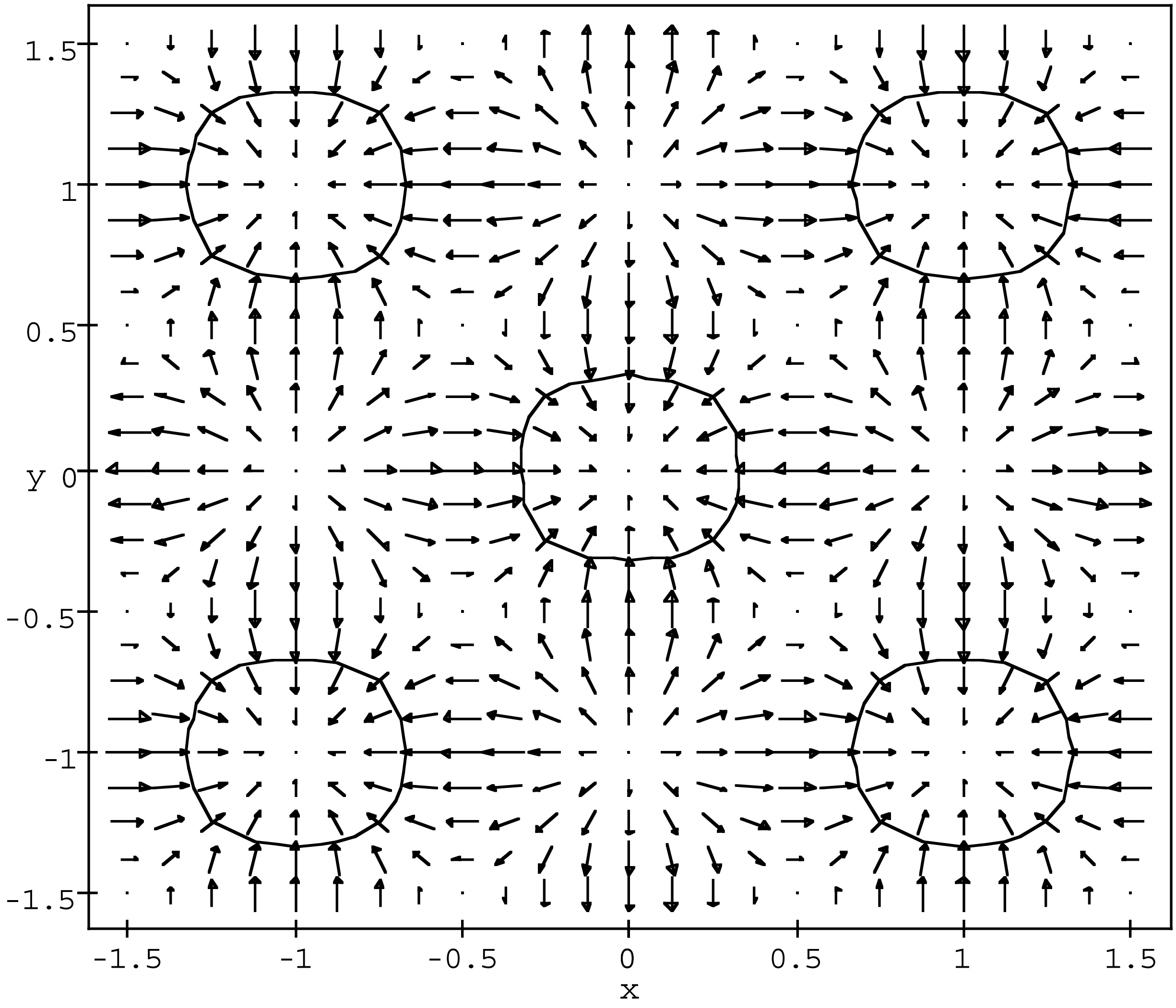
Exercise 1: Based on the plot below, circle each of the critical points and label them by type.



Exercise 2: Based on the plot below, circle each of the critical points and label them by type.



Exercise 3: Based on the plot below, circle each of the critical points and label them by type.



Solving for critical points algebraically

Students often have trouble finding critical points algebraically because they are not used to solving pairs of non-linear equations. In this section we describe three techniques which will prevent you from losing critical points.

Technique #1 All Equations Factor

Example 1: Find the critical points of $f(x, y) = xy(x + y - 1)$.

Solution: We first take the partial derivatives and set them equal to zero. It turns out to make the rest of the problem easier, if we do not multiply out the factors of f but instead use the product rule. If we do this we get:

$$\frac{\partial f}{\partial x} = y(x + y - 1) + xy$$

$$\frac{\partial f}{\partial y} = x(x + y - 1) + xy$$

The next step is to set both partials equal to zero, and factor them as far as is possible. This gives:

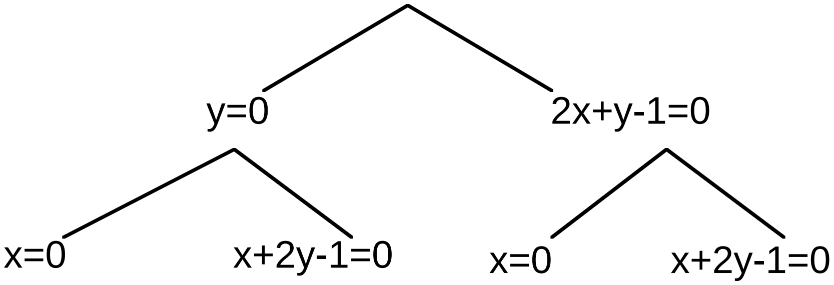
$$\frac{\partial f}{\partial x} = y(2x + y - 1) = 0$$

$$\frac{\partial f}{\partial y} = x(x + 2y - 1) = 0.$$

At each critical point, at least one factor from each equation must be zero!

With this in mind, we can make a diagram to keep track of the roots.

We write down each of the factors of the first equation. Under each factor, we write down the factors of the second equation. If we do this we get:



Our diagram will help us keep track of the different ways we can take one factor from the first equation and another factor from the second equation. Choose any path from the top of the diagram to the bottom, picking up any equations you run into. There are four different paths, giving the following four sets of equations:

$y = 0$	$y = 0$	$2x + y - 1 = 0$	$2x + y - 1 = 0$
$x = 0$	$x + 2y - 1 = 0$	$x = 0$	$x + 2y - 1 = 0$

In this example solving each set of equations will give us one critical point, so we see that the four critical points are: $(0,0)$, $(1,0)$, $(0,1)$, $(1/3,1/3)$.

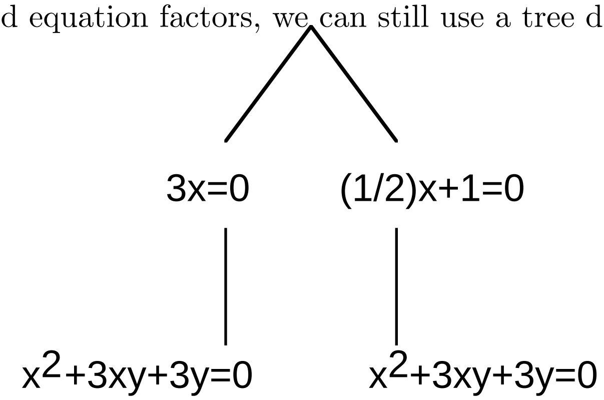
Technique #2 One Equation Factors

Example 2: Find the critical points of $f(x,y) = x^3/3 + (3/2)x^2y + 3xy$
Calculating the partials gives

$$\frac{\partial f}{\partial x} = x^2 + 3xy + 3y = 0$$

$$\frac{\partial f}{\partial y} = (3/2)x^2 + 3x = 3x((1/2)x + 1) = 0.$$

Because the second equation factors, we can still use a tree diagram; we get:



So if $x = 0$ then $3y = 0$, so $(0,0)$ is a critical point.

If $(1/2)x + 1 = 0$ then $x = -2$; substituting this into the other equation gives $4 - 6y + 3y = 4 - 3y = 0$ so $y = 4/3$, and the other critical point is $(-2, 4/3)$.

Technique #3 No Equation Factors, But a Combination of the Two Equations Does Factor

Example 3: Find the critical points of $f(x,y) = 6xy - (1/3)(x+y)^3$
Calculating the partials gives

$$\frac{\partial f}{\partial x} = 6y - (x+y)^2 = 0$$

$$\frac{\partial f}{\partial y} = 6x - (x+y)^2 = 0.$$

Here neither equation factors; but if we subtract the second equation from the first we get:

$$6y - 6x = 0$$

so

$$x = y.$$

You can then plug this into either of the two original equations to get the solution.