

Math U242, Spring 2005, Quiz #2, Solutions

1. (5 points) $\int \frac{x^3 + 2x}{1 + x^2} dx$

$$x^3 + 2x = (1 + x^2)(ax + b) + r = ax + b + ax^3 + bx^2 + r \Rightarrow \begin{matrix} a = 1 \\ b = 0 \end{matrix} \Rightarrow x^3 + 2x = x + x^3 + r \Rightarrow r = x.$$

Hence, $x^3 + 2x = (1 + x^2) \cdot x + x$ and $\frac{x^3 + 2x}{1 + x^2} = x + \frac{x}{1 + x^2}$. Substituting this equality into the original integral we obtain

$$\int \frac{x^3 + 2x}{1 + x^2} dx = \int x dx + \int \frac{x}{1 + x^2} dx = \boxed{\frac{x^2}{2} + \int \frac{x}{1 + x^2} dx + C}.$$

To calculate $\int \frac{x}{1 + x^2} dx$ we use the substitution $u = 1 + x^2$ so that $du = 2x dx$. Then

$$\int \frac{x}{1 + x^2} dx = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln|u| + C = \boxed{\frac{1}{2} \ln(1 + x^2) + C}.$$

Hence,

$$\int \frac{x^3 + 2x}{1 + x^2} dx = \frac{x^2}{2} + \int \frac{x}{1 + x^2} dx + C = \boxed{\boxed{\frac{x^2}{2} + \frac{1}{2} \ln(1 + x^2) + C}}.$$

2. (5 points) $\int \frac{2x - 2}{x^2 - 2x - 3} dx$

$$\boxed{x^2 - 2x - 3 = (x - 3)(x + 1)}.$$

$$\frac{2x - 2}{x^2 - 2x - 3} = \frac{A}{x - 3} + \frac{B}{x + 1} = \frac{A(x + 1) + B(x - 3)}{x^2 - 2x - 3}.$$

Therefore,

$$2x - 2 = A(x + 1) + B(x - 3) = (A + B)x + (A - 3B).$$

Hence,

$$\begin{aligned} A + B &= 2 & B &= 2 - A \\ A - 3B &= 1 & \Rightarrow A - 3(2 - A) &= -2 \Rightarrow 4A - 6 = -2 \Rightarrow A = 1 \Rightarrow B = 1. \end{aligned}$$

$$\boxed{\frac{2x - 2}{x^2 - 2x - 3} = \frac{1}{x - 3} + \frac{1}{x + 1}}$$

and

$$\int \frac{x^3 + 2x}{1 + x^2} dx = \int \frac{dx}{x - 3} + \int \frac{dx}{x + 1} = \boxed{\ln|x - 3| + \ln|x + 1| + C}.$$