Analysis 2. Spring 2008
Homework 1. Due on February 4

Problem 1. Let X and Y be finite dimensional complex vector spaces, $\dim X = n$, $\dim Y = m$. Let $A : X \to Y$ be a linear map. Given a basis $\mathbf{e} = \{e_1, \dots, e_n\}$ of X and a basis $\mathbf{f} = \{f_1, \dots, f_m\}$ we denote by

$$[A]_{\mathbf{f}}^{\mathbf{e}} \;=\; \{A^i_j\}$$

the matrix of A corresponding to these base. In other words, the matrix $\{A^i_j\}$ is defined by

$$A\Big(\sum_j x^j e_j\Big) \;=\; \sum_{i,j} (A^i_j\, x^j)\, f_i.$$

Recall that for $x \in X$ we denote by

$$[x]_{\mathbf{e}} \;=\; \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix}$$

the column of coordinates of x with respect to the basis $\mathbf{e}$. Similarly, for $y \in Y$ we denote by $[y]_{\mathbf{f}}$ the coordinates of y with respect to the basis $\mathbf{f}$. Then we have

$$x \;=\; \mathbf{e}\cdot[x]_{\mathbf{e}} \;=\; (e_1, \dots, e_n)\begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix}, \qquad y \;=\; \mathbf{f}\cdot[y]_{\mathbf{f}}, \qquad [A]_{\mathbf{f}}^{\mathbf{e}}\,[x]_{\mathbf{e}} \;=\; [Ax]_{\mathbf{f}}.$$

Notice that in the equality $[A]_{\mathbf{f}}^{\mathbf{e}}\,[x]_{\mathbf{e}} \;=\; [Ax]_{\mathbf{f}}$ the index $\mathbf{e}$ in the left hand side appears once as a subscript and once as a superscript. Then it disappears in the right hand side, as it should according to the Einstein convention. Also $\mathbf{f}$ is a subindex on both sides of the equation. This is a general rule – a subscript can be cancelled out by a superscript, but it can not jump up.

Let $\widetilde{\mathbf{e}} = \{\widetilde{e}_1, \dots, \widetilde{e}_n\}$ be another basis of X and let $\tilde{\mathbf{f}} = \{\tilde{f}_1, \dots, \tilde{f}_m\}$ be another basis of Y. We denote by $S_{\mathbf{e}}^{\widetilde{\mathbf{e}}}$ and $S_{\mathbf{f}}^{\tilde{\mathbf{f}}}$ the transition matrices such that

$$(\widetilde{e}_1, \dots, \widetilde{e}_n) \;=\; (e_1, \dots, e_n)\, S_{\mathbf{e}}^{\widetilde{\mathbf{e}}}, \qquad (\tilde{f}_1, \dots, \tilde{f}_m) \;=\; (f_1, \dots, f_m)\, S_{\mathbf{f}}^{\tilde{\mathbf{f}}}.$$

Let

$$[A]_{\tilde{\mathbf{f}}}^{\widetilde{\mathbf{e}}} \;=\; \{\widetilde{A}^i_j\}$$

the matrix of A corresponding to the base $\widetilde{\mathbf{e}}$ and $\tilde{\mathbf{f}}$, i.e.,

$$A\Big(\sum_j x^j \widetilde{e}_j\Big) \;=\; \sum_{i,j} (\widetilde{A}^i_j\, x^j)\, \tilde{f}_i.$$

a. Find $[A]_{\tilde{\mathbf{f}}}^{\widetilde{\mathbf{e}}}$ in terms of $[A]_{\mathbf{f}}^{\mathbf{e}}$, $S_{\mathbf{e}}^{\widetilde{\mathbf{e}}}$, and $S_{\mathbf{f}}^{\tilde{\mathbf{f}}}$.

b. Consider the special case when $Y = \mathbb{C}$ and $\mathbf{f} = \tilde{\mathbf{f}}$ is the standard basis of $\mathbb{C}$, i.e., $f_1 = 1 \in \mathbb{C}$. Notice that in this case $A : X \to \mathbb{C}$. In other words $A \in X^*$, where X^* is the *dual space* to X, i.e., $X^* = \mathrm{Hom}(X, \mathbb{C})$. Thus A is a *linear functional* on X, i.e., a *covector*. Write all the formulas of part (a) in this case.

c. The dual map to A is the map $A^* : Y^* \to X^*$, defined as follows. Let $\mu \in Y^*$ be a linear functional on Y. We define the linear functional $A^*\mu$ on X by

$$(A^*\mu)(x) \ := \ \mu(Ax), \qquad x \in X.$$

Let $\{e^1, \ldots, e^n\}$ be the basis of X^* dual to $\mathbf{e}$. Recall that it is defined by the formula

$$e^j(e_k) \ = \ \begin{cases} 1, & j = k; \\ 0, & j \neq k. \end{cases}$$

Similarly, let $\{f^1, \ldots, f^m\}$ be the basis of Y^* dual to $\mathbf{f}$. Find the matrix

$$[A^*]^{\{f^1,\ldots,f^m\}}_{\{e^1,\ldots,e^n\}}$$

of the operator A^* corresponding to these base in terms of the matrix $[A]^{\mathbf{e}}_{\mathbf{f}}$.

d. Let $\{\tilde{e}^1, \ldots, \tilde{e}^n\}$ and $\{\tilde{f}^1, \ldots, \tilde{f}^m\}$ be the base dual to $\tilde{\mathbf{e}}$ and $\tilde{\mathbf{f}}$ respectively. Express the matrix

$$[A^*]^{\{\tilde{f}^1,\ldots,\tilde{f}^m\}}_{\{\tilde{e}^1,\ldots,\tilde{e}^n\}}$$

in terms of $[A^*]^{\{f^1,\ldots,f^m\}}_{\{e^1,\ldots,e^n\}}$, $S^{\mathbf{e}}_{\tilde{\mathbf{e}}}$, and $S^{\mathbf{f}}_{\tilde{\mathbf{f}}}$.

2. If $f : \Omega \to \mathbb{R}^m$, where Ω is open in $\mathbb{R}^n$, and if u is a vector in $\mathbb{R}^n$, the *directional derivative* of f at $p \in \Omega$ in the direction of u is defined as

$$D_u f(p) \ := \ \lim_{t \to 0+} \frac{f(p + tu) - f(p)}{t}.$$

Sometimes we also use notations $D_u f(p) = uf(p) = \frac{\partial f}{\partial u}(p)$. Notice that $D_u f(p)$ is a vector in $\mathbb{R}^m$.

Remark: It is very instructive to consider first the special case when $m = 1$. In this case $D_u f(p)$ is a number.

a. Show that if f is differentiable at p, then $D_u f(p)$ exists for every $u \in \mathbb{R}^n$, and

$$D_u f(p) \ = \ df_p \cdot u$$

(in the right hand side we multiply the matrix df_p and the vector u).

b. By considering the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(s, t) \ = \ \frac{s^2 t}{s^4 + t^2}$$

for $(s, t) \neq 0$, and $f(0, 0) = 0$, show that f need not be differentiable at p even if $D_u f(p)$ exists for every unit vector u.

c. Suppose Ω is convex. Show that if $\|df_p\| < c$ for all $p \in \Omega$, then for every $x_1, x_2 \in \Omega$ we have

$$|f(x_1) - f(x_2)| \ < \ c\,|x_1 - x_2|.$$

Hint: Use the restriction of f to the line connecting x_1 and x_2 and the mean value theorem for a function of one variable.

Problem 3. Show that if U is open in $\mathbb{R}^n$ and $f : U \to \mathbb{R}^n$ is continuously differentiable, with df_p nonsingular for every $p \in U$, then $f(U)$ is open.