Two Ways of Looking at a Newtonian Supertask

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Introduction

In his paper "A Beautiful Supertask," (Laraudogoitia 1996), Perez Laraudogoitia (PL) described a dynamical system, which we will call ST, comprised of an infinite number of particles in a bounded region of space that can interact only via Newtonian elastic collisions. In the initial state, one particle moves towards the others which are stationary. In the final state, all the particles are stationary. Applying the principle of time reversal invariance, PL asserted that the stationary system could spontaneously self-excite, yielding a system in which particles are moving.

In detail, particles labeled P_1, P_2, $P_3, \ldots$, each of mass 1, are initially at rest at positions $x_1 = 1/2$, $x_2 = 1/4$, $x_3 = 1/8, \ldots$. An additional particle P_0, also of mass 1, is initially ($t = 0$) at position $x = 1$ and moves leftward towards P_1 with velocity -1. At time $t = 1/2$, P_0 collides with P_1. We assume that each particle in the system obeys Newton's three laws of motion and that each collision is perfectly elastic. Consequently, at $t = 1/2$ when P_0 collides with P_1 at $x_1 = 1/2$, P_0 stops and P_1 begins to move towards P_2 with velocity -1. At time $t = 3/4$, P_1 collides with P_2. P_1 stops at $x_2 = 1/4$, while P_2 moves toward P_3 with velocity -1. This process continues ad infinitum.

What happens at $t = 1$? By that time, every particle P_i has collided with its immediate left-hand neighbor and so is stationary. Therefore, all the particles are stationary. In this final state, the momentum and energy are both zero. However, before $t = 1$, the momentum was equal to -1 and the energy was $1/2$. Although each collision conserves momentum and energy, the entire process conserves neither.

Now consider the same set of particles whose initial positions are the same as they were in ST. Pick some arbitrary time, t_0, and assume that at this time P_j is moving towards the right with velocity $+1$ and that all the other particles are stationary. This is a possible motion of the system since it is obtained by applying the time reversal operator to one of the states of ST. Extrapolating backwards in time, we see that at some time δ_t earlier than t_0, the system was stationary. Thus we have self-excitation; a stationary system has spontaneously evolved into a moving one. In addition, the self-excitation can occur at any time so we have indeterminism. Let us call this scenario, based on the time reversed motion of ST, TRST.

Alper and Bridger (AB) were intrigued by these supertasks and sought to determine the source of the seemingly paradoxical behavior of ST and TRST. They suggested (Alper, Bridger 1998) that the "local," i.e., particle-by-particle, analysis used to derive the properties of these systems is too weak to uncover the origins of that behavior. By using a "global" analysis, based on embedding the system of particles in a Hilbert space with its associated norm, they showed that singularities arise in Newton's laws as applied to these supertasks. In view of these singularities, they described such supertasks as non-Newtonian.

In an ensuing series of papers, PL and AB have each attempted to address the critiques of the other (Alper, Bridger 1999, Laraudogoitia 1999, 200?). In order to terminate this potentially infinite process (after all, the infinities in ST and TRST are more than sufficient!), the three authors agreed to prepare a single paper which would provide a neutral discussion of ST, TRST, and related supertasks and would also delineate the areas of agreement and disagreement. A model for this type of paper has been provided by a recent paper which discussed other aspects of infinite Newtonian systems (Alper, Bridger, Earman, Norton, 2000).

Basic Definitions and Principles

In this section we make explicit the definitions and principles that are used in the analysis of ST and TRST in order to clarify the subtleties in mathematics that lead to different interpretations of such terms as "continuous", "differentiable", and "Newtonian".

We restrict our discussion to the types of systems required for the analysis of ST and TRST. Each system consists of a set of particles each of mass m. We set $m = 1$ for convenience. The motion of each particle is governed by Newton's laws of motion. We assume that there are no external forces. The particles are restricted to move in one dimension. They interact by means of perfectly elastic collisions so that each collision conserves kinetic energy as well as momentum. The initial state of each of these systems is either the state in which all the particles are stationary or the state in which all the particles are stationary except for one that moves with constant velocity. The principles are as follows:

(P1) If at time t, a particle is moving at constant velocity, it will continue to move at that same velocity until it collides with another particle. (Law of Inertia).

(P2) If particle P_i lies strictly to the left of particle P_j at some time t_0, then for all times t, both $t > t_0$ and $t < t_0$, P_i will never be strictly to the right of P_j. In other words a particle cannot pass through another. (Law of Impenetrability).

(P3) If particle P_i is moving with constant velocity v and collides with a stationary particle P_j, P_i will become stationary and P_j will move with velocity v. (Law of Collision).

(P4a) For all t, the position of particle P_i at time t, denoted by $x_i(t)$, is a continuous function of t. (Law of Continuity).

(P4b) For all t, the position of particle P_i at time t, denoted by $x_i(t)$, is a twice differentiable function of t. We denote this second derivative by $x_i''(t)$. (Law of Differentiability).

(P5) A particle P_i is said to *disappear* at time t^* if $x_i(t)$ is well-defined for $t < t^*$ but, for any position a, the assertion $x_i(t^*) = a$ leads to a contradiction.

A few remarks may help clarify these principles. **(P1)**, the Law of Inertia, is just Newton's first law of motion. **(P2)**, the Law of Impenetrability, is required if **(P3)** is to be true. **(P3)** is a consequence of Newton's second and third laws of motion (as applied to elastic collisions of particles) together with **(P2)**. The application of Newton's laws to the elastic collision of a stationary with a moving particle has two solutions. The particles can exchange velocities as stated in **(P3)**, or each particle can maintain the velocity it had before the collision. This latter solution is excluded by **(P2)**.

PL and AB agree that for any isolated collision the discontinuities in velocity and acceleration of the two particles at the instant they collide is unimportant. (By an isolated collision we mean one such that there is an open 2-dimensional region defined by an open interval of time and an open interval of distance such that this two-dimensional region includes the collision but does not contain any of the other collisions.) The discontinuities resulting from a collision can be treated mathematically by either using distribution functions, in particular the Dirac delta function (Messiah, 1999), or by using smoothed force and velocity functions as suggested by Friedberg and discussed in detail by Grünbaum (Grünbaum 1970). Thus, we can ignore the apparent singularities arising at the moment of any isolated collision. Then, Newton's second law, $x_i'' = F_i$, where F_i is the force exerted on P_i, must hold for each particle. Thus **(P4b)** is satisfied. Since every differentiable function is continuous, **(P4a)** holds as well.

Although **(P5)** seems to be out of place, since nothing like it appears in Newtonian mechanics, this assumption is needed by both PL and AB in their analyses of some of the more extreme consequences of dynamical supertasks that are discussed below. It is important to note that "disappearance" must be distinguished from "indeterminateness". Disappearance means that the particle is nowhere, i.e., it no longer exists. Indeterminateness, on the other hand, means that the particle may be somewhere, but we cannot determine what its position is. For example, suppose a particle's position as a function of time takes the form

For t in one of the intervals $[0, \frac{1}{2})$, $[\frac{3}{4}, \frac{7}{8})$, $[\frac{15}{16}, \frac{31}{32}) \ldots$, $x(t) = 0$.

For t in the remaining intervals $[\frac{1}{2}, \frac{3}{4})$, $[\frac{7}{8}, \frac{15}{16})$, $[\frac{31}{32}, \frac{63}{64}) \ldots$, $x(t) = 1$.

What is $x(1)$? We don't know, but we can choose its position to be either 0 or 1 without the possibility of contradiction. Consequently, we say that the particle's position is indeterminate; we do not say that the particle has disappeared. [1] In a later section, we shall describe some examples of "disappearance".

The disagreement: What is a Newtonian System?

PL and AB do not agree on the appropriate definition of "Newtonian" for the analysis of supertasks like ST and TRST. Consider a system of particles together with the forces acting on them. This system is said to be "Local Newtonian", if the motion of each particle can be described by the solution of Newton's second law applied to that

[1] This example is a variation of the argument used by Benacerraf in his critique of Thomson's Lamp (Benacerraf 1970).

particle: $x_i''(t) = F_i(\{x(t)\})$, where $\{x(t)\}$ denotes the positions of all of the particles at time t. The system is "Globally Newtonian", if there exists a solution to the entire set of differential equations in a mathematical sense to be described below.

AB maintain that "Newtonian" should mean "Global Newtonian" because the global condition maintains the primacy of the laws of conservation of momentum and energy. In the global analysis, Newton's laws break down for those systems such as ST and TRST in which the conservation laws fail.

PL agrees that a suitable global analysis is preferable to a local one, but believes AB's proposed solution is not successful because it (i) excludes from analysis certain unbounded but otherwise unobjectionable systems and (ii) rejects as non-Newtonian certain bounded systems that are locally Newtonian and which, in fact, do satisfy both energy and momentum conservation. These issues will be discussed in detail after we describe AB's global embedding.

Embeddings in Banach Space

AB introduced three infinite-dimensional vector spaces that contain the position, velocity, and force vectors, respectively. Typical vectors in each of these spaces take the form

$$X(t) = \big[x_1(t), x_2(t), \ldots, x_n(t), \ldots\big], \tag{1}$$

$$V(t) = \big[v_1(t), v_2(t), \ldots, v_n(t), \ldots\big], \tag{2}$$

$$F(X(t)) = [F_1(X(t)), F_2(X(t)), \ldots, F_n(X(t)), \ldots]. \tag{3}$$

The force vector is *not* an explicit function of t. Such an explicit dependence on time would indicate the presence of external forces which we have assumed to be absent.

Instead, the force on any particle at time t is a function of the positions of all the particles at time t.

Newton's equations of motion can always be written in the form of two coupled first order differential equations:

$$V(t) = X'(t); \quad F(X(t)) = V'(t). \tag{4}$$

The definition of the derivative in these equations requires some care. For an ordinary real valued function $g(t)$, the usual freshman calculus definition suffices:

$$g'(t) = \lim_{h \to 0} \frac{g(t+h) - g(t)}{h}. \tag{5}$$

For vector-valued functions, $G(t) = \big[g_1(t), g_2(t), \ldots, g_i(t) \ldots\big]$ we can define

$$G'(t) = \lim_{h \to 0} \frac{G(t+h) - G(t)}{h}. \tag{6}$$

But what is meant by "$\lim_{h\to 0}$"? For scalar functions Q we say that the real number $L = \lim_{h\to 0} Q(h)$ if the difference $D = Q(h) - L$ can be made as "small" as we wish by setting h sufficiently close (but not equal) to 0. Since D is a real number, we can measure "smallness" by computing $|D|$, the absolute value of D.

For vector valued functions, both $Q(h)$ and L are vectors and so is D. The length of D is defined by a function called the norm of D and denoted by $\|D\|$. Although all norms must satisfy certain mathematical properties (modeled after the properties of absolute value), there is, in general, no unique norm for a vector space. However, for any legal norm, the definition of the derivative given above is meaningful, although it is possible that a function may be differentiable (have a derivative) in one norm, but not in another. An equivalent (implicit) definition of $G'(t)$, in which the norm appears explicitly, takes the form:

$$\lim_{h \to 0} \left\| G'(t) - \frac{G(t+h) - G(t)}{h} \right\| = 0. \tag{7}$$

For a finite n-dimensional vector z, one frequently encounters the n-dimensional Pythagorean norm:

$$\|z\|_{\text{Pyth}} = \sqrt{\sum_{i=1}^{n} z_i^2} \tag{8}$$

where z_i are the n components of z.

For infinite-dimensional vectors, AB had, in their previous work, used an extension of the Pythagorean norm called the "ℓ^2" norm" defined by:

$$\|z\|_2 = \sqrt{\sum_{i=1}^{\infty} z_i^2}. \tag{9}$$

Systems in which the norm of any of the vectors in equations 1–3 is infinite are excluded from analysis.

As PL pointed out, the ℓ^2 norm defined by eq. 9. introduces several problems. First, although ST can be analyzed using this norm, many very similar systems are excluded. For example, the system in which the particles are at positions $x_i = 1/\sqrt{i}$ is excluded because the ℓ^2 norm of the position vector is not finite. Second, the ℓ^2 norm as defined in eq. 9 is not compatible with Galilean invariance. Consider two Galilean inertial frames moving with constant velocity with respect to each other. An infinite-dimensional velocity vector with finite norm in the first frame will, when transformed to the second frame, have infinite norm.

Many of these difficulties are remedied by introducing the "Sup" norm. This norm is defined by

$$\|z\| = \sup\{|z_i|\},\ i = 1,\ 2,\ 3, \ldots \tag{10}$$

where "sup" denotes the *supremum* or *least upper bound.*

Although the ℓ^2 norm of a vector such as $[1,\ 1,\ 1, \ldots]$ is infinite, the Sup norm of the same vector is equal to 1. Galilean invariance is automatic. Consider the vector obtained by adding the constant velocity vector $\mathbf{w} = [c,\ c,\ c, \ldots]$ to a velocity vector v. Then $\|v+w\| \le \|v\| + |c|$, a finite number. However, the embedding using the Sup norm cannot be applied to unbounded systems because such systems involve vectors whose Sup norms are infinite.

In the global analysis of ST using the Sup norm, the position, velocity, and force vectors, $\mathbf{X}(t)$, $\mathbf{V}(t)$, and $\mathbf{F}\big(\mathbf{X}(t)\big)$, respectively, are all admissible vectors. Every relevant vector has finite norm because, at any time, its components are bounded. (The components of $\mathbf{X}$ and $\mathbf{V}$ are all bounded by 1; note also that there are at most 2 non-zero components of $\mathbf{V}$ and $\mathbf{F}$ at any given time.)

The singularity in ST arises as follows: For $t < 1$, $\|\mathbf{X}'(t)\| = 1$, but for $t = 1$, $\|\mathbf{X}'(t)\| = 0$. Thus, the global velocity vector, $\mathbf{V}(t) = \mathbf{X}'(t)$, is discontinuous at $t = 1$, since a discontinuity in the norm of a vector function implies a discontinuity in the vector function itself. Since differentiability implies continuity, the global velocity vector is not differentiable. Thus Newton's equations of motion become singular at $t = 1$. It is important to note that this singularity cannot be removed by using distribution functions or by smoothing. A more mathematically detailed analysis of this discontinuity is provided in the Appendix.

Infinite-dimensional vector spaces that possess a norm (and are complete) are called Banach spaces. AB called their three spaces Hilbert spaces, when they used the ℓ^2 norm, because this norm arises from an "inner product," a defining characteristic of Hilbert spaces.

The Crux of the Disagreement

PL has proposed an example, which we call **2D**, that illustrates the fundamental difference between the local and global analysis. In **2D**, the particles, lying in the plane, are arranged in odd-even pairs, each pair at a different horizontal level above the x-axis. At $t = 0$, P_{2n-1}, the odd-numbered particle of each pair, begins to move rightward at unit velocity towards P_{2n}, its even-numbered partner.

$$
\begin{array}{rcl}
P_1 = \left(\frac{1}{2}, \frac{1}{2}\right) & \longrightarrow & P_2 = \left(1, \frac{1}{2}\right) \\
P_3 = \left(\frac{1}{3}, \frac{2}{3}\right) & \longrightarrow & P_4 = \left(1, \frac{2}{3}\right) \\
P_5 = \left(\frac{1}{4}, \frac{3}{4}\right) & \longrightarrow & P_6 = \left(1, \frac{3}{4}\right) \\
\cdots & \longrightarrow & \cdots \\
P_{2n-1} = \left(\frac{1}{n+1}, \frac{n}{n+1}\right) & \longrightarrow & P_{2n} = \left(1, \frac{n}{n+1}\right) \\
\cdots & \longrightarrow & \cdots
\end{array}
$$

Or, graphically: [diagram]

Let us denote by $v_n(t)$ the velocity of the nth particle (with the vertical component, always equal to 0, suppressed). At time $t = 0$, $v_{2n-1} = 1$ and $v_{2n} = 0$.

In each horizontal track, one and only one collision occurs in the (closed) time interval $[0, 1]$. However, for any small $\epsilon > 0$, the total number of collisions occurring in the time interval $[t - \epsilon, t]$ increases as t approaches 1.

An infinite number of collisions occur in any time interval $[1 - \epsilon, 1]$.

Unlike ST and TRST, **2D** is free of paradoxes; both energy and momentum are conserved, there is no self-excitation, and particles are neither created nor destroyed. Furthermore, from the local point of view, the collisions of **2D** are equivalent to those of ST, and **2D** is, of course, locally Newtonian.

PL and AB agree that in the absence of smoothing, **2D** is singular in the global view. We can write the velocity vector in the form $\mathbf{V}(t) = \big[v_1(t), v_2(t), v_3(t), \ldots\big]$, where $v_n(t)$ is the horizontal component of the velocity of particle n at time t. In the global embedding, $\mathbf{V}(t)$ is discontinuous at $t = 1$. The proof is easy.

Look at $\mathbf{V}(t)$ at $t = 1 - \epsilon$ and at $t = 1$, where again $\epsilon > 0$ is some arbitrarily small number.

$$\mathbf{V}(1) = [0,\ 1,\ 0,\ 1,\ 0,\ 1, \ldots,\ 0,\ 1,\ 0,\ 1,\ 0,\ 1, \ldots]$$

$$\mathbf{V}(1-\epsilon) = [\underbrace{0,\ 1,\ 0,\ 1,\ 0,\ 1, \ldots,}_{\text{finitely many agree}}\ \underbrace{1,\ 0,\ 1,\ 0,\ 1,\ 0,\ \ldots}_{\text{infinitely many disagree}}]$$

No matter how small ϵ gets, $\|\mathbf{V}(1) - \mathbf{V}(1-\epsilon)\|$ is always 1. Thus, $\mathbf{V}(1-\epsilon)$ does not approach $\mathbf{V}(1)$ as $\epsilon \to 0$, and so $\mathbf{V}(t)$ is discontinuous at $t = 0$.

Despite this global discontinuity, PL regards this system as nonpathological and completely amenable to analysis using the local method. The system consists of a countably infinite number of independent two-body systems. PL believes that the global method, which excludes this perfectly reasonable system, is far too restrictive.

The introduction of smoothing has a dramatic affect on this analysis. Assumption **(P4b)**, the "Law of Differentiability," requires that velocity be a differentiable function of time. Suppose that when a particle P, moving at constant velocity v, is within a distance d of particle Q, P begins to slow down, while Q begins to speed up. Furthermore, suppose P's velocity becomes zero when it reaches the position formerly occupied by Q, at which time Q's velocity has reached the constant velocity v. To simplify the mathematics a bit, we assume that these changes in velocity are linear. These velocity functions are not differentiable; they have sharp corners. However, we can smooth out these corners by using quadratic functions instead of linear ones.

With smoothing, particle P_{2n-1} travels at constant unit velocity from position $1/(n+1)$ to $1-d$ and then slows down to velocity 0 during a time interval $2d$. The velocity function for the odd-numbered particles is given by:

$$v_{2n-1}(t) = \begin{cases} 1, & \text{for } t \in \left[0, 1 - \frac{1}{n+1} - d\right] \\ 1 - \frac{1}{2d}\left(t - 1 + d + \frac{1}{n+1}\right), & \text{for } t \in \left[1 - \frac{1}{n+1} - d, 1 - \frac{1}{n+1} + d\right] \\ 0 & \text{for subsequent times} \end{cases} \tag{11}$$

Consequently, for times $s, t > 1 - \frac{1}{n+1} - d$,

$$|v_{2n-1}(s) - v_{2n-1}(t)| \leq \frac{|s-t|}{2d}. \tag{12}$$

A similar inequality holds for the velocities of the even-numbered particles.

Eq. 12 demonstrates that if the interaction distance d is constant (the same for each of the particles) then the velocity vector $\mathbf{V}(t) = [v_1(t), v_2(t), v_3(t), \dots]$ is a uniformly continuous function of t. With quadratic smoothing, $\mathbf{V}(t)$ becomes differentiable. Thus, the smoothed **2D** is globally Newtonian, and is as acceptable a system to AB as it is to PL.

Suppose on the other hand, that the interaction distance d decreases to zero as the particle number $n \to \infty$. This model can be realized by assuming that the particles are elastic spheres such that particles P_{2n} and P_{2n-1} have diameter 10^{-n}. This smoothed system, which like the unsmoothed **2D** is locally Newtonian but globally non-Newtonian, is acceptable to PL but not to AB.

It is informative to compare **2D** with ST. In ST, there are an infinite number of particles in a bounded one-dimensional interval. It is evident that in this case, d must be a decreasing function of particle number n; there is simply no room for the interaction distance to remain constant. Thus, as we have previously seen, ST is locally Newtonian, but globally non- Newtonian.

For **2D** on the other hand, it is not obvious whether or not d can be held constant. Although there is "plenty of room" along each horizontal track, the distance between horizontal tracks decreases as a function of n. If we allow the particles to have extension in only one dimension, then we can ignore interactions between horizontal tracks. The distance parameter d can be constant and the system is globally Newtonian. If, however, we regard the particles as three-dimensional, then their sizes must decrease as a function of n, and presumably d will also be a function of n. In this scenario, **2D** is not globally Newtonian.

For AB, any dependence of the range of interaction, d, on the specific pair of particles interacting is unnatural. This behavior manifests itself in a singularity in the global analysis of the system. Thus, AB classify the system as non-Newtonian. PL notes that both the smoothed and unsmoothed versions of **2D** are perfectly amenable to the local analysis and moreover, do not violate the conservation laws that AB consider central. Consequently, in PL's opinion, the global analysis, as currently formulated, is too restrictive.

Pushing the Envelope: Even Wilder Supertasks

Now that we have what both PL and AB believe to be correct descriptions of the local and global formulations, we present two supertasks, more extreme than even ST and TRST, whose local analyses shed further light on what can happen.

In (Laraudogoitia 1998), PL proposed what might be called a "super-supertask" (SST). He considered two ST systems that are mirror images of each other, i.e., the particles are initially at $\pm\left(\frac{1}{2^i}\right)$. These systems spontaneously self-excite in such a way as to produce, in finite time, a system whose particles oscillate at speeds that approach infinity. At a critical time, the position functions $x_i(t)$ of these particles become discontinuous. PL takes this violation of principle **(P4a)** to indicate that these particles have disappeared. By invoking time reversal invariance, PL argues that particles spontaneously appear, i.e, "Creation ex Nihilo". This supertask is perfectly legitimate according to the local description.

AB have also constructed an example to be analyzed using the local method, in which a particle disappears. In contrast to PL's example, which depends on the violation of continuity, AB's example is based on a violation of the impenetrability of particles **(P2)**.

Consider the set of stationary particles P_i at positions $\frac{1}{2^i}$, $i = 1, 2, 3, \ldots$, just as in ST. Now imagine particle P_0 approaching this system from the left (rather than from the right). In particular, assume that at $t = 0$, P_0 is at $x = -1$ and is moving to the right with velocity $+1$. We further assume that no self-excitation of the initially stationary set of particles occurs in the time interval $[0, 1]$. This absence of self-excitation is certainly a possible evolution of the system of initially stationary particles. The effect of this assumption can also be provided by the somewhat weaker assumption that no particle moves unless it is either P_0 or else is involved in a chain of collisions at least one of which involves P_0. We now show that immediately after $t = 1$, P_0 has disappeared.

At time $t = 1$, nothing special happens. P_0 is at $x = 0$, but since none of the other particles are either at $x = 0$ or to the left of $x = 0$, there are no collisions. Now choose some time t_1, $t_1 > 1$. Where is P_0 at this time? Suppose it is at $x = a$, where $0 < a$. We can find an integer j such that $\frac{1}{2^j} < a$. But then, by the principle of impenetrability, particle P_0 must have either collided with P_j before P_0 reached $x = a$ or else P_0 collided with some other particle which is to the left of P_j. In either case, P_0 stops moving before it reaches $x = a$. Thus, P_i is not at $x = a$. It is equally clear that it can't be at any other position $a \leq 0$ either. Since the only restriction on t_1 is that it is greater than 1, assumption **(P5)** tells us that P_0 must vanish at all times after $t = 1$.

The Fundamental Indeterminacy of ST

Whose analysis of ST is correct? To answer this question, we note that in his analysis of supertasks, John Norton (Norton 1999) has given an explicit example of a nonsingular classical dynamical system that satisfies Newton's equations but nevertheless violates determinism and energy conservation. The indeterminism arises because there is no last particle. Consequently, there are fewer equations than particle coordinates describing the system.

In the spirit of Norton's work, we propose the following local analysis of ST that exposes the fundamental indeterminacy of the system. Two alternative arguments are given: the first returns PL's momentum nonconservation result; the second results in momentum conservation.

We first introduce some notation:

1. $p_i(0)$ is initial momentum of particle P_i.

2. $p_i(1)$ is the momentum of P_i at $t = 1$.
3. ΔpA_i is the change of momentum of P_i in a type A collision — a collision that sets P_i in motion.
4. ΔpB_i is the change of momentum of P_i in a type B collision — a collision that returns P_i to its stationary state.

We assume that the momentum of a particle can change only as the result of a collision. Note that in the original analysis of ST, the *total* momentum appears to change at time $t = 1$, when no collision is taking place.

Argument 1: Non-conservation of momentum. First we compute the total momentum at $t = 0$:

$$p_0(0) = -1;$$

$$p_i(0) = 0 \quad \text{(for i not equal 0);}$$

$$p_{\text{total}}(0) = -1 \quad \text{(the total momentum at } t = 0.)$$

Now we compute the total momentum at $t = 1$, noting that because the motion is leftward, setting a particle in motion corresponds to a negative momentum change.

$$
\begin{aligned}
\Delta p A_0 &= 0 \quad \text{(since there is no collision that sets } P_0 \text{ in motion);} \\
\Delta p A_i &= -1 \quad \text{(for i not equal 0);} \\
\Delta p B_i &= +1 \quad \text{(for all i).} \\
p_{\text{total}}(1) &= (\text{total momentum at } t = 1) \\
&= \sum_{i=0}^{\infty} p_i(1) \\
&= \sum_{i=0}^{\infty} p_i(0) + \Delta p A_i + \Delta p B_i \\
&= (-1 + 0 + 1) + (0 + (-1) + 1) + (0 + (-1) + 1) + \cdots \\
&= 0 + 0 + 0 + \cdots \\
&= 0 \\
&\neq p_{\text{total at } t\,=\,0}.
\end{aligned}
$$

Argument 2: Conservation of momentum.

As in Argument 1, $p_{\text{total at } t\,=\,0} = -1$.

Now we compute the momentum at $t = 1$ after regrouping the summands.

$$\begin{aligned}
p_{\text{total at } t\,=\,1} &= \sum_{i=0}^{\infty} p_i(1) \\
&= \sum_{i=0}^{\infty} p_i(0) + \Delta p B_i + \Delta p A_{i+1} \\
&= p_0(0) + (\Delta p B_0 + p_1(0) + \Delta p A_1) + (\Delta p B_1 + p_2(0) + \Delta p A_2) + \cdots \\
&= -1 + (+1 + 0 + (-1)) + (+1 + 0 + (-1)) + \cdots \\
&= -1 + 0 + 0 + \cdots \\
&= -1 \\
&= p_{\text{total at } t\,=\,0}.
\end{aligned}$$

Which argument is correct: Argument 1 that conserves neither energy or momentum or Argument 2 which conserves both? The overall ordering of the terms in both sums is well-defined; it corresponds to the temporal order of the progression of ST. Does the strangeness of ST arise because a nonconvergent sum can have different values when the summands are grouped differently? In this view, the conflict between the arguments is simply a signal of (algebraic) indeterminism. Or, does the strangeness arise because ST is inherently ill-defined? We are not sure. We might note that the *Cesaro* sum of this series is 0, a result in agreement with PL's finding of non-conservation of momentum. [2]

Argument 2 maintains momentum conservation; the momentum at $t = 1$ is equal to -1. What carries this momentum? Consider, once again, AB's supertask in which a particle approaches from the left. After the completion of this supertask, assume a self-excitation à la TRST. We then have the following scenario: A particle, P^*, approaches

[2] The Cesaro sum of an infinite series is defined by

$$\sum_{i=0}^{\infty} (\text{Cesaro})\ a_i = \lim_{n\to\infty} \left(\frac{1}{n+1}\right) \sum_{i=0}^{n} S_i \ \text{ where } S_i = \sum_{j=0}^{i} a_j$$

from the left (moving rightward) and disappears at $t = 1$. Then self-excitation occurs, resulting in a sequence of collisions of particles with their right-hand neighbors. Let ST$'$ be the entire process, beginning with the particle P^* approaching from the left. Now let TRST$'$ be the time reversal of ST$'$. In TRST$'$, a particle approaches from the right (moving leftward) and ST occurs as before. However, unlike ST, at the completion of TRST$'$ an unnamed particle appears at the origin at $t = 1$, moving toward the left. This particle carries the momentum and the energy. In this scenario, energy and momentum are conserved. Unfortunately, at the present time we do not know how or even whether this process can be analyzed from the global standpoint.

Conclusion

AB have argued that energy and momentum conservation are such fundamental physical laws that any violation of them requires explanation. The mathematical analysis using the global approach of ST results in singularities, suggesting to AB that the supertasks are not Newtonian systems. PL counters by noting that because of its limitations, the global approach is incapable of analyzing many infinite systems of interest. In particular, it rejects some systems that are not only amenable to the local analysis but that, in addition, conserve momentum and energy.

Setting this disagreement aside, both PL and AB agree that the global approach may well offer a fruitful move towards understanding and possibly eliminating the pathological behavior of infinite dynamical systems. In the analysis of ordinary classical mechanical systems, the approach using the energy and momentum conservation rules is often more convenient than solving the equations of motion because the conservation approach has the merits of avoiding the difficult details of the solution. Perhaps that is the case for infinite systems as well.

The supertask TRST′ may be a first step toward an energy analysis of Newtonian supertasks. Unlike ST which violates energy conservation, the locally Newtonian TRST′ maintains energy conservation even though it results in the disappearance of a particle. We have discussed in detail the fundamental indeterminacy in the analysis of ST. A further study of TRST' and its relationship with ST may clarify this uncertainty and the still murky analysis of Newtonian supertasks.

APPENDIX: In ST, $\mathbf{V}(t)$ is neither continuous nor differentiable at $t = 1$.

Consider the function $\|\mathbf{V}(t)\|$ which is the composition of the function $\mathbf{V}(t)$ and the norm function $x \to \|x\|$. Since the composition of continuous functions is continuous, it will suffice to show that the norm is continuous but the composition isn't. The norm satisfies the triangle inequality: $\|A+B\| \leq \|A\| + \|B\|$. Applying this to the vector sums $\mathbf{v} = (\mathbf{v}-\mathbf{w}) + \mathbf{w}$ and $\mathbf{w} = (\mathbf{w}-\mathbf{v}) + \mathbf{v}$, and using $\|\mathbf{v}-\mathbf{w}\| = \|\mathbf{w}-\mathbf{v}\|$, we see that $-\|\mathbf{v}-\mathbf{w}\| \leq \|\mathbf{v}\| - \|\mathbf{w}\| \leq \|\mathbf{v}-\mathbf{w}\|$. Thus, when $\mathbf{v}$ and $\mathbf{w}$ are close, so are $\|\mathbf{v}\|$ and $\|\mathbf{w}\|$. This makes the norm continuous. It therefore remains to show that the composition $\|\mathbf{V}(t)\|$ is discontinuous at $t = 1$.

For any time t^* before $t = 1$ there is always a later time t_n, $t^* < t_n < 1$ and a particle P_n which is moving with velocity 1 at $t = t_n$ (i.e. $v_n(t_n) = 1$). Thus, $\|\mathbf{V}(t_n)\| = 1$, so

$$\lim_{t \to 1} \|V(t)\|, \text{ if it exists, must be 1.}$$

On the other hand, $\|\mathbf{V}(1)\| = \|0\| = 0$. Since the value of the function at 1 is not equal to its limit, the function $\|\mathbf{V}(t)\|$ fails to be continuous at $t = 1$, so $\mathbf{V}(t)$ also fails to be continuous. A classical result of analysis states that differentiability implies continuity. Thus, the failure of $\mathbf{V}(t)$ to be continuous at $t = 1$ means it can not be differentiable at $t = 1$ either.

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