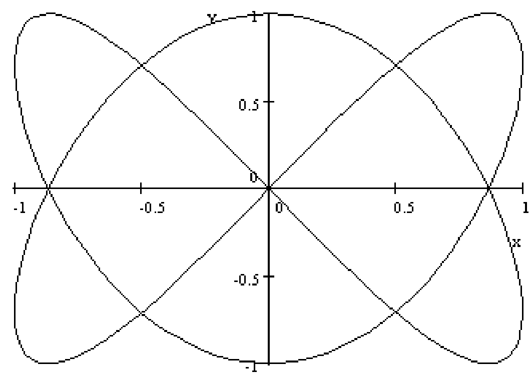
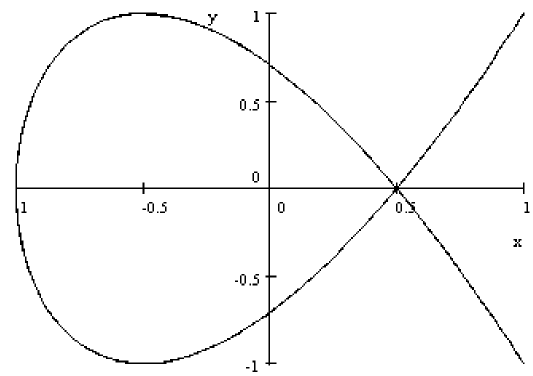
Project: Lissajous Figures

A parametric plot of the form $(\sin(nt+a), \cos(kt+b))$ is called a *Lissajous curve,* named after Jules-Antoine Lissajous, a 19th century mathematician who studied them. (You can also use cos and sin, or two sines or two cosines — for example $(\sin(nt+a), \sin(kt+b))$. We will examine only the cases $a = b = 0$. Here are a few examples:



Type I: $(\sin(2t), \cos(3t))$, $0 \leq t \leq 2\pi$



Type II: $(\cos(2t), \cos(3t))$, $0 \leq t \leq 2\pi$

Basic Lissajous Curves For each pair of numbers (n, k) there correspond four basic Lissajous curves:

$$(n,k) \rightarrowtail \begin{cases} (\sin nt, \cos kt) \\ (\cos nt, \cos kt) \\ (\sin nt, \sin kt) \\ (\cos nt, \sin kt) \end{cases}$$

Type I versus Type II Note that Lissajous curves come in two "Types". Type I curves are "smooth", while Type II curves have sharp "ends".

Exercise A Use a calculator or computer to plot the two remaining Lissajous curves for $(2, 3)$ to go with the two plotted above. (Plot over the interval $0 \leq t \leq 2\pi$.) How many look like Type I? Type II? Include labeled sketches of these plots in your project.

Exercise B Try some more; for example, look at the 4 curves corresponding to $(4, 5)$. Also, $(3, 5)$ and $(3, 6)$. Do you think that for each (n, k) there will always be curves of both types? Include these plots in your project, noting which are which.

The Fundamental Period Notice that both of the figures above have been plotted over the interval $0 \leq t \leq 2\pi$. In the first one, you start when $t = 0$, at the point $(\sin 0, \cos 0) = (0, 1)$. As t increases, you start tracing the curve moving leftward etc. You continue until $t = 2\pi$, when you are at $(\sin 2\pi, \cos 2\pi) = (0, 1)$ again. You pass through every point on the curve once, and end up where you started. You might say that 2π is the *fundamental period* of this curve: if you take a domain smaller than $[0, 2\pi]$ — for example, $[0, 3\pi/2]$ —

you won't get the whole curve; if you take a bigger domain — e.g. $[-\pi/4, 2\pi]$, or $[0, 5\pi/2]$ — you will trace some points more than once. *Not every Lissajous curve has fundamental period equal to* 2π. For example, $(\sin(6t), \cos(8t))$ has fundamental period $\pi/2$; in other words, plotting this for $0 \leq t \leq \pi/2$ gives the whole curve, but no smaller interval will.

Below you will find more exercises related to Lissajous curves. To answer them, you will want to make lots of plots and try to understand how the curves are traced out. You can plot using Matlab (see sample for doing this at the end), but it is probably best to use your graphing calculator, since it draws curves so slowly that you can see how they are formed as t varies; you can also usually see when a section is being retraced especially (see brief instructions at the end).

Helpful Fact Keep in mind that the functions $y = \sin bx$ and $y = \cos bx$ repeat exactly every $2\pi/b$ radians. This is also expressed by saying that $\sin bx$ and $\cos bx$ have period $2\pi/b$.

Exercises C

1. Explain why the fundamental period of a Lissajous curve is never bigger than 2π. (See the **Helpful Fact** above.)
2. What are the fundamental periods for the four Lissajous curves corresponding to $(2, 3)$? (HINT: They will be of the form $2\pi/Q$ for some whole number Q .) The best way to do this is to try different values of Q, starting with $Q = 1$, and check for the biggest one which gives the whole curve.
3. What are the fundamental periods of the four curves corresponding to $(4, 6)$? $(6, 9)$? What is your guess for $(2d, 3d)$ where d is any whole number? Do all these curves "look" the same?
4. (This applies to smooth, i.e. Type I curves.) If the curve $(\sin(nt), \cos(kt))$ is to repeat itself in some interval $0 \leq t \leq 2\pi/d$, *both* $\sin(nt)$ *and* $\cos(kt)$ must go through some multiples of their periods in this interval. Write a paragraph justifying this assertion. So suppose that this curve repeats itself after $2\pi/d$ radians. How is $2\pi/d$ related to the periods $2\pi/n$ and $2\pi/k$ of the sine and cosine functions? Note that $2\pi/d$ is smallest when d is biggest. What must d be in order that $2\pi/d$ be a *fundamental* period?

NOTE A Type II curve is drawn from one "end" to the other. Thus, it seems to have only half the fundamental period of the corresponding smooth (Type I) curve. If you want, you can think of the "complete" or "closed" Type II curve as going from one "end" to the other, and then back to the beginning. In this case it will have the same fundamental period as the corresponding smooth ones.

5. **Summary:** Suppose d divides both n and k, so $n = n'd$ and $k = k'd$. How are the curves corresponding to (n, k) related to the curves corresponding to (n', k') (appearance and fundamental periods)? When will the Type I curves corresponding to (n', k') have period exactly 2π?

Extra Credit (requires some vector calculus) If you think of the curve as the path of a particle moving through space as time t varies, then the particle is always moving in the same direction for the smooth or Type I curves, while there has to be a reversal of direction at the "ends" in the Type II curves. The *velocity* becomes 0 at these points. If you know some vector calculus, see what this would mean in each of the four cases corresponding to (n, k). (You may assume $\gcd(n, k) = 1$.)

HINT: Suppose we are dealing with the $(\sin nt, \cos kt)$ case. Then the velocity vector is simply the pair of derivatives $(n \cos nt, -k \sin kt)$. If this becomes the zero vector $(0, 0)$, then both components $n \cos nt$ and $-k \sin kt$ have to be zero for the same value of t. But the zeros of cos are the numbers $\pi/2 + P \cdot \pi$, while the zeros of sin are the numbers $Q \cdot \pi$ (P and Q are whole numbers). So

$$\begin{aligned} nt &= \pi/2 + P \cdot \pi \\ kt &= Q \cdot \pi \end{aligned}$$

Solving for t in each of these, we get: $\dfrac{\pi/2 + P \cdot \pi}{n} = \dfrac{Q \cdot \pi}{k}$ (since they both equal t). Use this to show that k *must be even.*

Now use this same idea to see when you get Type II curves in the other three cases corresponding to (n, k).

PLOTTING IN MATLAB: To plot the parametric curve $\langle \sin(3t), \cos(5t) \rangle$ over the interval $0 \leq t \leq 2\pi$, simply type:

```
ezplot('sin(3*t)','cos(5*t)',[0,2*pi])
```

at the MatLab >> prompt. Be careful not to forget the `' '` signs around the functions! Ezplot does not give you much control over the appearance of the plot (e.g. you can't set the color). You can use the plot command for more flexibility, but it is best in an m-file. For example, to produce the same plot but in red, you can create the m-file:

```
t = [0:0.01:2*pi];
x=sin(3*t);
y=cos(3*t);
plot(x,y,'r')
```

(You can even put multiple plots in the same window using the "`hold on`" command.)

PLOTTING ON A TI CALCULATOR: Put your calculator into *parametric* and *radian* mode by pressing the `MODE` button and highlighting these choices. Press the `Y=` button and enter the functions `cos(3T)` and `sin(5T)`. Finally, press the `WINDOW` button and choose `T` to go from 0 to 2π, and x and y to go from -1 to 1; now `GRAPH`.