Facts About Visible Points

(For reference, see "Some Number Theory Notes" on our homepage.)

Definition: *The* ***fundamental lattice*** $\mathcal{L}$ *consists of all those points in the plane with integer coordinates. These points are called lattice points.*

For example, $(2, 3)$ and $(-7, 14)$ are points in $\mathcal{L}$, while $(1, 4/5)$ does not lie in $\mathcal{L}$.

Suppose that a tree is growing at each point in $\mathcal{L}$. These trees are infinitely thin, but opaque. Imagine that we are standing at the origin and we look toward the tree at $(6, 8)$. Can we see it? The answer is no, because our line of sight from $(0, 0)$ to $(6, 8)$ passes through the *nearer* point $(3, 4)$, and the opaque tree there obscures it. On the other hand, we *can* see the tree at $(3, 4)$ because it's the closest lattice point on this line to the origin. This inspires the following definitions.

Definition: *The* **line to point** P *means the line from the origin to point* P.

Definition: *A point* P *in the lattice* L *is called* **visible** *if the line to* P *contains no lattice point closer to the origin than* P.

Remark: In giving the mathematical definition of "visible" we have dispensed with the "tree" metaphor, since it was just used as motivation. It is also easier to say "a point P is visible" than to say "the tree at point P is visible."

It is not too difficult to determine, using number theory, exactly which points are visible. This is done in the exercises. We will use the following facts from number theory:

Very Important Theorem: *If* $d = \gcd(a, b)$ *then there are integers* x *and* y *such that* $d = xa + yb$.

(This Theorem is proved using the Euclidean algorithm to find x and y. See "GCDs" on the website.)

We sometimes express this by saying that "The greatest common divisor of a and b is a linear combination of a and b.Here is a very useful consequence of this theorem:

Proposition: If $a \mid b \cdot c$ and $\gcd(a, b) = 1$, then $a \mid c$.

Proof: Since $1 = \gcd(a, b)$, we can write $1 = a \cdot x + b \cdot y$ (because of the Very Important Theorem). Multiply through by c: $c = a \cdot x \cdot c + (b \cdot c) \cdot y$. Since a divides everything on the right-hand side, it must divide c. ■

Here, then, is the answer about Visibles:

Theorem: (m, n) is visible if and only if $\gcd(m, n) = 1$.

Proof: There are two things to prove since this and an "if and only if" statement: we must show that if (m, n) is visible, then $\gcd(m, n) = 1$, *and* we must show that if $\gcd(m, n) = 1$ then (m, n) is visible.

OK, so suppose (m, n) is visible, and let $d = \gcd(m, n)$. The m/d and n/d are whole numbers — since d goes evenly into a and b. (NOTE: Don't confuse the *division* A/B with the statement $A \mid B$, which says "A divides B."). Note that (m, n) and $(m/d, n/d)$ lie on the same line through the origin. But the point $(m/d, n/d)$ is closer to the origin than (m, n) unless $d = 1$. Since (m, n) is supposed to be visible, there can't be any point on its line closer to the origin, so we must have $d = 1$. This finishes the (easier) first half.

Now let's suppose $\gcd(m, n) = 1$. We will show that (m, n) is visible by showing that any other point on its line through the origin is further away. Suppose (a, b) also lies on this line. Then $\frac{n}{m} = slope = \frac{b}{a}$. Cross multiplying, we have $mb = an$. Since $\gcd(m, n) = 1$, we must have $m \mid a$, so $a = km$. Substituting, we have: $mb = an = (km)n$, so $b = kn$. Therefore $(a, b) = (km, kn)$, so unless $k = 1$, the point (a, b) lies further from the origin than (m, n). Thus, (m, n) is visible.

This completes the proof, which we indicate by the symbol: ■