

The Irrationality of $\sqrt{2}$

Some basic facts

1. **Definition:** A number X is rational if it can be written as a quotient of two whole numbers: $X = A/B$. If X is rational then, by reducing to lowest terms, we can assume that A and B have no common factor. In particular, **if X is rational, we can assume that $X = A/B$ where A and B are not both even.**
2. If N is even, its square N^2 is also even, while if N is odd, its square is odd. This shows that **if the square of a number is even, it must have been even.**

So now we come to the proof. Suppose that $\sqrt{2}$ is rational. We will show that this leads to an absurdity, so can not be true.

If $\sqrt{2}$ is rational, we can write:

$$\sqrt{2} = A/B \text{ where } A \text{ and } B \text{ are whole numbers and } \mathbf{\text{not both even.}}$$

We can now square both sides of this equation, obtaining:

$$\begin{aligned} (\sqrt{2})^2 &= \frac{A^2}{B^2} \\ 2 &= \frac{A^2}{B^2} \\ 2B^2 &= A^2. \end{aligned}$$

Thus, since A^2 is twice a number, A^2 is even, so we can conclude **A is even**, and we can write: $A = 2M$ for some whole number M . Thus, we can rewrite our equation:

$$2B^2 = A^2 = (2M)^2 = 4M^2.$$

Now, we can divide through by 2, obtaining:

$$B^2 = 2M^2 \text{ so } B^2 \text{ is even.}$$

But, since B^2 is even, **B is even also**. Thus, we have seen that **both A and B turn out to be even**. This contradicts the fact that we chose A and B so as not to have any common factors.

Thus, the only assumption we made, namely that $\sqrt{2}$ is rational, leads to the contradiction that A and B have no common factors, yet both are even. Thus, $\sqrt{2}$ can not be rational.

Q.E.D. (= *Quod Erat Demonstrandum* = Which Was to be Proved)