Final Review

The final will be cumulative, covering issues of the course going back to the first day. In particular, you are expected to know basic facts about the equation of a line, especially the point-slope formula, $y = m(x - x_0) + y_0$. You should understand the various aspects of slope (formula, geometric, rate-of-change). Know the quadratic formula for solving $ax^2 + bx + c = 0$.

It is essential that you understand the **definition** of the derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

All applications and many important techniques go back to this.

Be able to use the definition to work out the derivatives of low order polynomials, and functions such as simple polynomial, $\sqrt{x}$, or $1/(ax+b)$. Part of what you are being tested on with such questions is your algebraic skill.

Know how to use the rules of differentiation (the sum, product, quotient, and chain rules) together with the derivatives in the following list, to differentiate complicated functions such as $\sin(3x^2)e^{\tan(4x)}$ or $9\ln(2^x + 5)/(x^2 + e^x)$.

Basic functions whose derivatives you must know:

$$x^n, \quad \frac{1}{x^n}, \quad \sqrt{x}, \quad a^x, \quad e^x, \quad \ln(x), \quad \sin(x), \quad \cos(x),$$
$$\tan(x), \quad \sec(x), \quad \arctan(x), \quad \arcsin(x)$$

Understand the geometry of derivatives: that $f'(x)$ is the slope of the tangent line to the graph of f at x, and that $f''(x)$ tells which way the graph bends. Use this to find maximum and minimum points of the function.

Understand that derivatives are *instantaneous* rates of change. This is a big piece of why they are important. This goes back to the definition of the derivative.

The derivative $\frac{dy}{dx}$—the instantaneous rate of change—and the difference quotient $\frac{\Delta y}{\Delta x}$—the average rate of change—*approximate each other*:

$$f'(x) = \frac{dy}{dx} \approx \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

This leads to the linear approximation formula:

$$f(x + \Delta x) \approx f(x) + f'(x)\Delta x.$$

Know the techniques of implicit and logarithmic differentiation, related rates, and how to find the derivative of various inverse functions. Be able to find asymptotes.

You should also know how to deal with parametric curves $x = x(t)$, $y = y(t)$, especially how to plot them on your calculator, the formula $\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$, and how to use this formula to find the equation of tangent lines and the places where these tangent lines are horizontal and vertical.

Whereas the derivative describes rates of change: $f'(x) = \frac{dy}{dx} \approx \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)}{\Delta x}$, the integral describes how to add up changes to reconstruct the function.

The Fundamental Theorem of Calculus is, as its name implies, *fundamental*. Know what it says. It enables us to compute some integrals in terms of our knowledge of derivatives. You should know how to integrate (i.e. antidifferentiate) functions like x^n $(n \neq -1)$, $1/x$, e^x, $1/(1+x^2)$ and $1/\sqrt{1-x^2}$, $\sin x$, $\cos x$ and $\sec^2 x$. Below are some sample problems.

Some Sample Problems

1. Use rules of differentiation to evaluate the derivatives of the following functions of x: $2^x \cos(x^3)$, $\ln(5x^7 \sin(3x))$, $\frac{3x^5 + 9}{4x^6 - 8}$, $x^3 e^{-3x}$, $x^3 e^{-x^2}$, $5\tan(\pi x) + 7\sec(5x)$, $\arctan(4x)$, $\arcsin(\sqrt{x})$.

2. For the function $f(x) = x^2 + \frac{3}{x+2}$, use the *definition* of the derivative to find $f'(x)$ (no formulas).

3. You are given the following information about a function f: Its value at $x = 2$ is 7 and its instantaneous rate of change at $x = 2$ is -4.

   (a) Use this information to find the equation of the tangent line at $x = 2$.

(b) Use part (a) to estimate $f(2.5)$.

(c) Suppose that $f''(x) < 0$ for all x. Is your estimate in part (b) an overestimate or underestimate (draw a diagram and explain).

4. Suppose $Q(x) = R(x)e^{-2x}$. If $R(1) = 2$ and $R'(1) = 5$, find $Q'(1)$.

5. The circumference $C = 2\pi R$ of a circle is measured to be 200 cm, with an error of ± 0.2 cm. If the Area $A = \pi R^2$ is computed using this value of the circumference, what is the maximum error? What will be the *relative* error?

6. y is implicitly defined to be a function of x by $y^3 + y = 6x$. One point on the graph of this function is $(x_0, y_0) = (5, 3)$.

   (a) Find the equation of the tangent line to the graph of y as a function of x at (x_0, y_0).

   (b) Use your answer to approximate y at $x = 5.21$. Give your answer as precisely as possible.

7. A point mass moves along the x-axis in such a way that its position at time t is $x(t) = e^{-t}\sin(t)$. By computing $x' = dx/dt$ and $x'' = d(x')/dt$, prove that this function $x(t)$ satisfies $x'' + 2x' + 2x = 0$.

8. Use logarithmic differentiation to differentiate the functions $f(x) = \dfrac{(2x+3)^9(5x+4)^7}{\sqrt{2x+\pi}\sin(x)}$ and $g(x) = (\tan(x))^x$.

9. Let $f(x) = x^3 + x^2 - x + 5$.

   (a) Find all critical points of this function, and determine which are maxima or minima.

   (b) Find all inflection points.

   (c) Draw a sketch of the curve $y = f(x)$ showing the maxima, minima and inflection points.

   (d) Give intervals where the function is increasing and where it is decreasing.

   (e) Find the absolute maximum and absolute minimum of $f(x)$ on the interval $[-2, 1]$.

10. If $g(x) = \dfrac{x}{x^2+4}$ then $g'(x) = \dfrac{(x^2+4)\cdot 1 - x(2x)}{(x^2+4)^2} = \dfrac{4-x^2}{(x^2+4)^2}$ and $f''(x) = \dfrac{-2x(4+x^2)(12-x^2)}{(x^2+4)^4}$. Use these to find all critical points of g, all inflection points. Does the curve $y = g(x)$ have any asymptotes? Draw a sketch of $y = g(x)$.

11. Let $\mathcal{C}$ be the curve with parametric equations $\begin{cases} x = t^2 \\ y = 3t^5 - 25t^3 + 60t \end{cases}$.

    (a) Find $\dfrac{dy}{dx}$ in terms of t.

    (b) Find all points where $\mathcal{C}$ has horizontal or vertical tangents.

    (c) Use your calculator to draw a sketch of $\mathcal{C}$ showing all interesting features (see part b); state what window you used.

12. Use Newton's method to find the smallest positive solution to $x^2 = \cos x$.

13. A closed, rectangular wooden box with square base and lid is to contain 128 cubic feet. The material for the base and the lid costs twice what the material for the sides costs. What are the dimensions of the least expensive box that can be made with these specifications?

14. Find the rectangle of maximum area, with a side lying along the x-axis, whose top vertices lie on the parabola $y = 8 - x^2$.

15. A certain coffee filter is in the shape of a cone of radius 20 cm and height 30 cm. When the depth of the coffee in the filter is 25 cm, coffee is draining *out* of it at a rate of 5 cc per minute. What is the rate of decrease of the depth of the coffee at that moment?

16. Using a subdivision of the interval $[1, 3]$ into 4 equal parts, and the evaluation set consisting of the left-hand end-points, calculate a Riemann sum that approximates $\displaystyle\int_1^3 \frac{x}{2x-1}\,dx$.

    (a) Is Riemann sum an overestimate or underestimate? Use a diagram to explain.

(b) Estimate the integral using 4 subdivisions, but this time use *midpoint rectangles*.

17. Evaluate the following integrals:

    (a) $\int \frac{2}{x} dx$
    (b) $\int \frac{5}{x^2} dx$
    (c) $\int 15 \sin x + 8 \cos x \; dx$
    (d) $\int \frac{10}{1+x^2} dx$
    (e) $\int 3e^x + e^{-2x} \; dx$
    (f) $\displaystyle\int \frac{5x + 2x^2 - 1}{\sqrt{x}} \; dx$
    (g) $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}} \; dx$

18. Calculate each of the following integrals by giving a *geometrical* argument.

    (a) $\int_{x=0}^{5} \sqrt{25 - x^2} \; dx$
    (b) $\int_1^3 5 + 3x \; dx$

19. Suppose that $G'(x) = g(x)$ for all x, and $g(1) = 7$, $g(3) = 5$, $G(1) = 2$ and $G(3) = 19$. Calculate $\int_1^3 g(x) \; dx$.

20. Calculate the definite integrals:

    (a) $\displaystyle\int_1^2 (3y^2 - \tfrac{1}{y}) \, dy$
    (b) $\displaystyle\int_0^{\pi/4} 3 \sec^2 x \, d$
    (c) $\displaystyle\int_0^1 \tfrac{x}{1+x^2} \; dx$

21. Calculate the derivative:

    (a) $f(x) = \int_0^x e^{t^2} \; dt$
    (b) $g(x) = \int_1^{\sqrt{x}} \sqrt{1+t^2} \, dt$

22. An object moves along the x-axis so that its acceleration at time t is given by $a(t) = e^t - t$ cm/s^2 If we know that its initial ($t = 0$) velocity is 7 cm/s and its initial position is at $x = 4$ cm, find its position at time t.