Answers to Some Sample Problems

1. Use rules of differentiation to evaluate the derivatives of the following functions of x

$$\begin{aligned}
\left[2^x \cos(x^3)\right]' &= 2^x\left[-\sin(x^3)3x^2\right] + [2^x \ln 2]\cos(x^3) \\
\left[\ln(5x^7 \sin(3x))\right]' &= [\ln(5) + 7\ln(x) + \ln\sin(3x)]' \\
&= 0 + \frac{7}{x} + \frac{\cos(3x)\cdot 3}{\sin(3x)} \\
\left[\frac{3x^5+9}{4x^6-8}\right]' &= \frac{(4x^6-8)[15x^4] - (3x^5+9)[24x^5]}{(4x^6-8)^2} \\
\left[x^3 e^{-3x}\right]' &= x^3\left[e^{-3x}\cdot(-3)\right] + e^{-3x}(3x^2) \\
\left[x^3 e^{-x^2}\right]' &= x^3[e^{-x^2}\cdot(-2x)] + e^{-x^2}\cdot[3x^2] \\
[5\tan(\pi x) + 7\sec(5x)]' &= 5\sec^2(\pi x)\cdot\pi + 7\sec(5x)\tan(5x)\cdot 5 \\
[\arctan(4x)]' &= \frac{4}{1+(4x)^2} \\
\left[\arcsin(\sqrt{x})\right]' &= \frac{\frac{1}{2}x^{-1/2}}{\sqrt{1-(\sqrt{x})^2}} = \frac{1}{2\sqrt{x}\sqrt{1-x}}
\end{aligned}$$

:

2. For the function $f(x) = x^2 + \dfrac{3}{x+2}$, use the *definition* of the derivative to find $f'(x)$ (no formulas).

$$\begin{aligned}
f(x+h) &= (x+h)^2 + \frac{3}{x+h+2} \\
f(x+h) - f(x) &= (x+h)^2 - x^2 + \frac{3}{x+h+2} - \frac{3}{x+2} \\
&= (x^2 + 2xh + h^2 - x^2) + \frac{3(x+2) - 3(x+h+2)}{(x+h+2)(x+2)} \\
&= 2xh + h^2 + \frac{-3h}{(x+h+2)(x+2)} \\
\frac{f(x+h)-f(x)}{h} &= 2x + h - \frac{-3}{(x+h+2)(x+2)} \xrightarrow[h\to 0]{} 2x - \frac{3}{(x+2)^2}
\end{aligned}$$

3. You are given the following information about a function f: Its value at $x = 2$ is 7 and its instantaneous rate of change at $x = 2$ is -4.

   (a) Use this information to find the equation of the tangent line at $x = 2$.
   Use point-slope $y - y_0 = m(x - x_0)$ with $y_0 = f(x_0) = f(2) = 7$ and $m = f'(2)$:

$$\begin{aligned}
y - 7 &= -4(x-2), \text{ or} \\
y &= -4(x-2) + 7
\end{aligned}$$

   (b) Use part (a) to estimate $f(2.5)$.
   $f(2.5) \approx y_{\tan} = -4(2.5-2) + 7 = -2 + 7 = 5$

   (c) Suppose that $f''(x) < 0$ for all x. Is your estimate in part (b) an overestimate or underestimate (draw a diagram and explain).
   Since $f''(x) < 0$ the curve is concave *down*, so its values lie below the tangent lines. Thus, the tangent line values are *greater*, so are overestimates.

4. Suppose $Q(x) = R(x)e^{-2x}$. If $R(1) = 2$ and $R'(1) = 5$, find $Q'(1)$.
   Product Rule: $Q'(x) = R(x)[e^{-2x}(-2)] + R'(x)e^{-2x}$, so

$$\begin{aligned}
Q'(1) &= R(1)\cdot(-2)e^{-2} + R'(1)e^{-2} \\
&= -4e^{-2} + 5e^{-2} = e^{-2} \approx 0.1353.
\end{aligned}$$

5. The circumference $C = 2\pi R$ of a circle is measured to be 200 cm, with an error of ± 0.2 cm. If the Area $A = \pi R^2$ is computed using this value of the circumference, what is the maximum error? What will be the *relative* error?

   Express A in terms of C: $R = \dfrac{C}{2\pi}$ so $A = \pi\left(\dfrac{C}{2\pi}\right)^2 = \dfrac{C^2}{4\pi}$. Note that $\dfrac{dA}{dC} = \dfrac{C}{2\pi}$

   The tangent-line approximation says that $\dfrac{\Delta A}{\Delta C} \approx \dfrac{dA}{dC}$ or $\Delta A \approx \dfrac{dA}{dC} \cdot \Delta C$ (here "Δ" is the error). Thus

$$\begin{aligned} \text{Error in } A &\approx \frac{C}{2\pi} \cdot \text{ (Error in } C) \\ \text{Error in } A &\approx \frac{200}{2\pi} \cdot (\pm 0.2) \approx \pm 6.3662 \end{aligned}$$

   Also,

$$\text{Relative Error in } A = \frac{\Delta A}{A} \approx \frac{\pm 6.3662}{200^2/(4\pi)} = 0.002 \text{ or } \tfrac{1}{5}\%.$$

6. y is implicitly defined to be a function of x by $y^3 + y = 6x$. One point on the graph of this function is $(x_0, y_0) = (5, 3)$.

   (a) Find the equation of the tangent line to the graph of y as a function of x at (x_0, y_0).
   We just need the derivative to apply point-slope:

$$\begin{aligned} 3y^2\frac{dy}{dx} + \frac{dy}{dx} &= 6 \\ \frac{dy}{dx} &= \frac{6}{3y^2+1} \\ &= \frac{6}{27+1} = \frac{3}{14}. \end{aligned}$$

   Thus:

$$\begin{aligned} y - 3 &= \tfrac{3}{14}(x-5) \text{ or} \\ y &= \tfrac{3}{14}(x-5) + 3 \end{aligned}$$

   (b) Use your answer to approximate y at $x = 5.21$. Give your answer as precisely as possible.

$$y \approx \tfrac{3}{14}(5.21 - 5) + 3 = 3.045$$

7. A point mass moves along the x-axis in such a way that its position at time t is $x(t) = e^{-t}\sin(t)$. By computing $x' = dx/dt$ and $x'' = d(x')/dt$, prove that this function $x(t)$ satisfies $x'' + 2x' + 2x = 0$.

   (This is straightforward differentiation, using the product rule, and algebra.)

8. Use logarithmic differentiation to differentiate the functions $f(x) = \dfrac{(2x+3)^9(5x+4)^7}{\sqrt{2x+\pi}\,\sin(x)}$ and $g(x) = (\tan(x))^x$.

   Take ln of both sides:

$$\ln(f(x)) = 9\ln(2x+3) + 7\ln(5x+4) - \frac{1}{2}\ln(2x+\pi) - \ln(\sin(x))$$

   Now differentiate:

$$\begin{aligned} \frac{f'(x)}{f(x)} &= \frac{9\cdot 2}{2x+3} + \frac{7\cdot 5}{5x+4} - \frac{1}{2x+\pi} - \frac{\cos x}{\sin x} \\ f'(x) &= \frac{(2x+3)^9(5x+4)^7}{\sqrt{2x+\pi}\,\sin(x)}\left(\frac{18}{2x+3} + \frac{35}{5x+4} - \frac{1}{2x+\pi} - \frac{\cos x}{\sin x}\right) \end{aligned}$$

   Do the same for $g(x)$:

$$\begin{aligned} \ln(g(x)) &= x\ln(\tan(x)) \\ \frac{g'(x)}{g(x)} &= x\frac{\sec^2(x)}{\tan(x)} + \ln(\tan(x)) \\ g'(x) &= (\tan(x))^x\left(x\frac{\sec^2(x)}{\tan(x)} + \ln(\tan(x))\right) \end{aligned}$$

9. Let $f(x) = x^3 + x^2 - x + 5$.

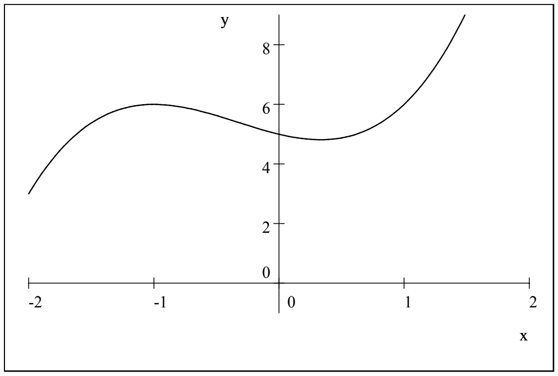
   (a) Find all critical points of this function, and determine which are maxima or minima.

   $f'(x) = 3x^2 + 2x - 1 = (3x-1)(x+1) \overset{\text{set}}{=} 0$, so $x = 1/3$ and $x = -1$ are the critical numbers, which give the points $(1/3, 130/27) \approx (0.33, 4.85)$ and $(-1, 6)$ having horizontal tangents.

   (b) Find all inflection points.

   $f''(x) = 6x + 2 \overset{\text{set}}{=} 0$, so $x = -1/3$ and $(-1/3, 146/27) \approx (-0.33, 5.41)$ is an inflection point (f'' clearly changes sign at $-1/3$).

   (c) Draw a sketch of the curve $y = f(x)$ showing the maxima, minima and inflection points.



   (d) Give intervals where the function is increasing and where it is decreasing.

   Function is decreasing between the two critical points; i.e. for $-1 < x < 1/3$; it is increasing elsewhere.

   (e) Find the absolute maximum and absolute minimum of $f(x)$ on the interval $[-2, 1]$.

   We need to test the critical points and the endpoints:

$$\begin{aligned} f(-2) &= 3 \\ f(-1) &= 6 \\ f(1/3) &= 130/27 \\ f(1) &= 6 \end{aligned}$$

   Thus, the absolute maximum is 6 and the absolute minimum is 3.

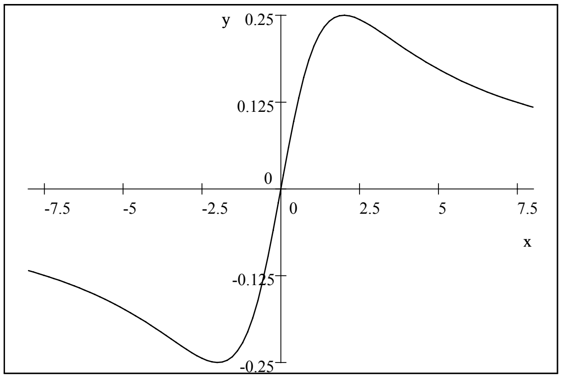
10. If $g(x) = \dfrac{x}{x^2+4}$ then $g'(x) = \dfrac{(x^2+4)\cdot 1 - x(2x)}{(x^2+4)^2} = \dfrac{4-x^2}{(x^2+4)^2}$ and $g''(x) = \dfrac{-2x(4+x^2)(12-x^2)}{(x^2+4)^4}$. Use these to find all critical points of g, all inflection points. Does the curve $y = g(x)$ have any asymptotes? Draw a sketch of $y = g(x)$.

    Note that $x^2 + 4$ is always positive, so never 0; thus, both $g(x)$ and its derivative are always defined. To find the critical points, set $g'(x) = 0$. This means $4 - x^2 = (2+x)(2-x) = 0$, so $x = \pm 2$ are the only critical points. When $x < -2$, $g'(x) < 0$; when x lies between -2 and 2, say $x = 0$, $g'(x) > 0$; when $x > 2$, $g'(x) < 0$. Thus, -2 is a relative minimum and 2 is a relative maximum. (You can also use the second derivative test: $g''(-2) > 0$ and $g''(2) < 0$.)

    To find inflection points, set $g''(x) = 0$; you get $x = 0$ and $x = \pm\sqrt{12}$. Since $x^2 + 4$ is never 0, there are no vertical asymptotes. On the other hand:

$$\lim_{x\to\infty} \frac{x}{x^2+4} = \lim_{x\to\infty} \frac{x/x^2}{x^2/x^2 + 4/x^2} = \lim_{x\to\infty} \frac{1/x\ (\to 0)}{1 + 4/x^2\ (\to 0)} = 0,$$

    so $y = 0$ is a horizontal asymptote. Here is a plot:



11. Let $\mathcal{C}$ be the curve with parametric equations $\begin{cases} x = t^2 \\ y = 3t^5 - 25t^3 + 60t \end{cases}$.

    (a) Find $\dfrac{dy}{dx}$ in terms of t.

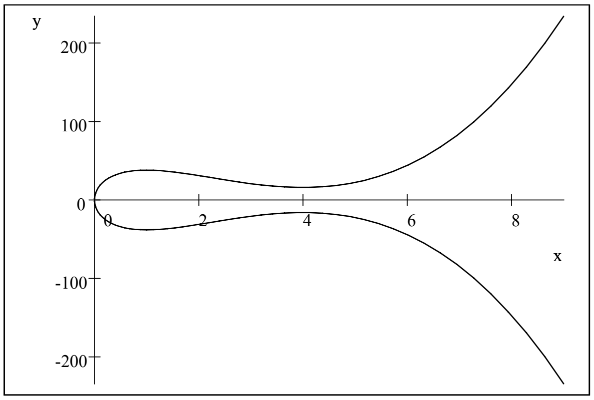
    (b) Find all points where $\mathcal{C}$ has horizontal or vertical tangents.

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy/dt}{dx/dt} = \frac{15t^4 - 75t^2 + 60}{2t} = \frac{15(t^4 - 5t^2 + 4)}{2t} \\ &= \frac{15(t^2 - 1)(t^2 - 4)}{2t} = \frac{15(t-1)(t+1)(t-2)(t+2)}{2t} \end{aligned}$$

    Thus, there are horizontal tangents at $t = \pm 1$, $t = \pm 2$, and a single vertical tangent at $t = 0$.

    (c) Use your calculator to draw a sketch of $\mathcal{C}$ showing all interesting features (see part b); state what window you used.

    $-3 \leq t \leq 3$:



12. Use Newton's method to find the smallest positive solution to $x^2 = \cos x$.

    We need to find a root of $x^2 - \cos x = 0$. A quick plot shows there is one between 1 and 2. Newton's method is the formula

$$x_{\text{new}} = x_{\text{old}} - \frac{f(x_{\text{old}})}{f'(x_{\text{old}})} = x_{\text{old}} - \frac{(x_{\text{old}})^2 - \cos(x_{\text{old}})}{2(x_{\text{old}}) + \sin(x_{\text{old}})}$$

    Starting with the guess $x = 1.5$, repeated iterations of the formula give guesses that stabilize at $x = 0.8241323123$.

13. A closed, rectangular wooden box with square base and lid is to contain 128 cubic feet. The material for the base and the lid costs twice what the material for the sides costs. What are the dimensions of the least expensive box

that can be made with these specifications?

$$\begin{aligned}
V &= s^2h = 128, \text{ so } h = 128/x^2. \\
\text{Cost of a side} &= sh \cdot (P) \\
\text{Cost of top or bottom} &= s^2 \cdot (2P) \\
\text{Total Cost} &= C = 4 \cdot sh \cdot P + 2s^2 \cdot 2P \\
C &= 4s(128/s^2)P + 4s^2P \\
&= 512P/s + 4s^2P
\end{aligned}$$

We find the critical point:

$$\begin{aligned}
C' &= -512P/s^2 + 8sP \overset{\text{set}}{=} 0 \\
512/s^2 &= 8s \\
s^3 &= 64 \\
s &= 4,\ h = 8.
\end{aligned}$$

14. Find the rectangle of maximum area, with a side lying along the x-axis, whose top vertices lie on the parabola $y = 8 - x^2$.

    Width $= W = 2x$, Height $= H = 8 - x^2$, so $A = 2x(8 - x^2) = 16x - 2x^3$.

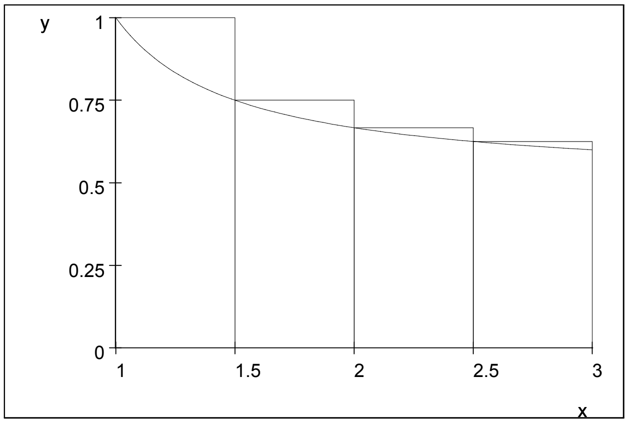
$$\begin{aligned}
A' &= 16 - 6x^2 \overset{\text{set}}{=} 0 \\
x^2 &= 16/6 = 8/3 \\
x &= \sqrt{8/3} \approx 1.6330 \\
W &\approx 3.2660,\ H \approx 5.3333, \text{ Area } \approx 17.4186.
\end{aligned}$$

15. A certain coffee filter is in the shape of a cone of radius 20 cm and height 30 cm. When the depth of the coffee in the filter is 25 cm, coffee is draining *out* of it at a rate of 5 cc per minute. What is the rate of decrease of the depth of the coffee at that moment?

    Let h be the coffee level and r the radius of the cone of coffee. From the usual similar triangles computation: $h/30 = r/20$, or $r = \frac{2h}{3}$. The volume V of the coffee is given by

$$\begin{aligned}
V &= \tfrac{1}{3}\pi r^2 h = \tfrac{1}{3}\pi\left(\tfrac{2h}{3}\right)^2 h = \tfrac{1}{3}\pi\tfrac{4h^2}{9}h \\
&= \tfrac{4\pi}{27} \cdot h^3. \\
\frac{dV}{dt} &= \tfrac{4\pi}{9} \cdot h^2\frac{dh}{dt} \\
-5\ c^3/m &= \tfrac{4\pi}{9}(25)^2\frac{dh}{dt}, \text{ so} \\
\frac{dh}{dt} &= \frac{-45}{4\pi \cdot 625} \approx -.005730\ c/m
\end{aligned}$$

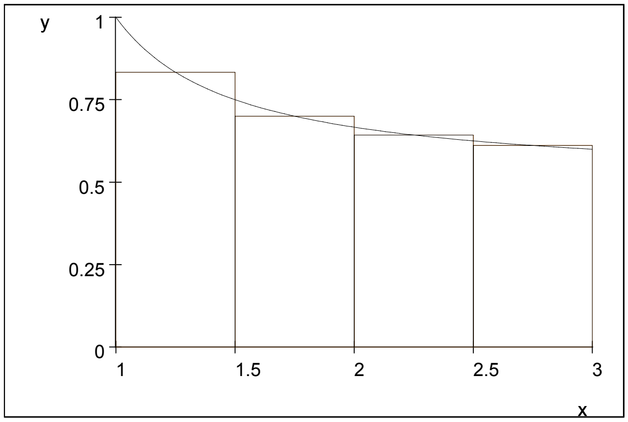
16. Using a subdivision of the interval $[1, 3]$ into 4 equal parts, and the evaluation set consisting of the left-hand end-points, calculate a Riemann sum that approximates $\displaystyle\int_1^3 \frac{x}{2x-1}\,dx$



$$\int_1^3 \frac{x}{2x-1}\,dx \approx 0.5\left(\frac{1}{2(1)-1}+\frac{1.5}{2(1.5)-1}+\frac{2}{2(2)-1}+\frac{2.5}{2(2.5)-1}\right) \approx 1.5208$$

(a) Is Riemann sum an overestimate or underestimate? Use a diagram to explain.
Since the curve is decreasing, the heights are too big, so it is an *overestimate.*

(b) Estimate the integral using 4 subdivisions, but this time use *midpoint rectangles.*



$$\int_1^3 \frac{x}{2x-1}\,dx \approx 0.5\left(\frac{1.25}{2(1.25)-1}+\frac{1.75}{2(1.75)-1}+\frac{2.25}{2(2.25)-1}+\frac{2.75}{2(2.75)-1}\right) = 1.3937$$

17. Evaluate the following integrals:

(a) $\int \frac{2}{x}dx = 2\int \frac{1}{x}dx = 2\ln x + C$

(b) $\int \frac{5}{x^2}dx = 5\int x^{-2}dx = 5(x^{-1}/(-1)) = -5x^{-1} + C$

(c) $\int 15\sin x + 8\cos x\ dx = -15\cos x + 8\sin x + C$

(d) $\int \frac{10}{1+x^2}dx = 10\arctan x + C$

(e) $\int 3e^x + e^{-2x}\ dx = 3e^x - \frac{1}{2}e^{-2x} + C$

(f) $\displaystyle\int \frac{5x+2x^2-1}{\sqrt{x}}\ dx = \int 5x^{1-1/2} + 2x^{2-1/2} - x^{-1/2}\ dx = \int 5x^{1/2} + 2x^{3/2} - x^{-1/2}\ dx$
$= 5x^{3/2}\cdot 2/3 + 2x^{5/2}\cdot 2/5 - x^{1/2}\cdot 2 = (10/3)x^{3/2} + (4/5)x^{5/2} - 2x^{1/2} + C$

(g) $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\ dx$. Let $u = \sqrt{x}$. Then $du = \frac{1}{2}x^{-1/2}dx$ or $\frac{dx}{\sqrt{x}} = 2du$. So $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\ dx = 2\int e^u\ du = 2e^u + C = e^{\sqrt{x}} + C$

18. Calculate each of the following integrals by giving a *geometrical* argument.

(a) $\int_{x=0}^{5} \sqrt{25-x^2}\ dx$
This is the area under the top half of the circle $x^2+y^2 = 15$, from 0 to 5; i.e. the area of 1/4 of the circle of radius 5. $\int_{x=0}^{5} \sqrt{25-x^2}\ dx = 25\pi/4$

(b) $\int_1^3 5+3x\ dx$.
This is the area under a trapezoid of width $3-1 = 2$, with bases (heights) $5+3 = 8$ and $5+9 = 14$: $\int_1^3 5+3x\ dx = \frac{1}{2}(8+14)\cdot 2 = 22.$

19. Suppose that $G'(x) = g(x)$ for all x, and $g(1) = 7$, $g(3) = 5$, $G(1) = 2$ and $G(3) = 19$. Calculate $\int_1^3 g(x)\ dx$.

By the Fundamental Theorem of Calculus, $\int_1^3 g(x)\ dx = G(3) - G(1) = 19 - 2 = 17.$

20. Calculate the definite integrals:

(a) $\displaystyle\int_1^2 (3y^2 - \tfrac{1}{y})\ dy = (y^3 - \ln y)\,|_1^2 = (8-\ln 2) - (1 - \ln 1) = 7 - \ln 2.$

(b) $\displaystyle\int_0^{\pi/4} 3\sec^2 x\, d = 3\tan x\,|_0^{\pi/4} = 3(1-0) = 3.$

(c) $\displaystyle\int_0^1 \frac{x}{1+x^2}\, dx$. Let $u = 1+x^2$. Then $du = 2xdx$ or $\frac{du}{2} = xdx$. Moreover $x = 0 \longrightarrow u = 1$ and $x = 1 \longrightarrow u = 2$. So $\int_0^1 \frac{x}{1+x^2}\, dx = 2\int_1^2 \frac{du}{u} = 2\ln u\,|_1^2 = 2\ln 2$.

21. Calculate the derivative:

(a) $f(x) = \int_0^x e^{t^2}\, dt$. $f'(x) = e^{t^2}|_{t=x} = e^{x^2}$.

(b) $g(x) = \int_1^{\sqrt{x}} \sqrt{1+t^2}\, dt$. $g'(x) = \sqrt{1+t^2}\|_{t=\sqrt{x}}(\sqrt{x})' = \sqrt{1+x}\,(\frac{1}{2}x^{-1/2}) = \frac{\sqrt{1+x}}{2\sqrt{x}}$.

22. An object moves along the x-axis so that its acceleration at time t is given by $a(t) = e^t - t$ cm/s^2 If we know that its initial ($t = 0$) velocity is 7 cm/s and its initial position is at $x = 4$ cm, find its position at time t.

Let $v(t)$ be the velocity at time t, and $x(t)$ the position at time t.

$a(t) = \dfrac{dv}{dt} = e^t - t$, so $v(t) = \int e^t - t\; dt = e^t - t^2/2 + C$. We are given $v(0) = 7$, so

$$\begin{aligned} 7 &= v(0) = e^0 - 0^2/2 + C = 1 + C \\ C &= 6 \\ v(t) &= e^t - t^2/2 + 6. \end{aligned}$$

Also, $v(t) = \dfrac{dx}{dt} = e^t - t^2/2 + 6$, so $x(t) = \int e^t - t^2/2 + 6\; dt = e^t - t^3/6 + 6t + D$. We are given $x(0) = 4$, so

$$\begin{aligned} 4 &= x(0) = e^0 - 0^3/6 + 6\cdot 0 + D \\ D &= 3. \end{aligned}$$

Thus, $x(t) = e^t - t^3/6 + 6t + 3$.