Laplace Methods for First Order Linear Equations

For first-order linear differential equations with *constant coefficients*, the use of Laplace transforms can be a quick and effective method of solution, since the initial conditions are built in. Here are some simple examples.

I. First order homogeneous

This is the simplest case: $ax' + bx = 0$ with initial condition $x'(0) = x_0$. First we divide through by a and let $k = b/a$; our equation becomes:

$$x' + kx = 0.$$

Now we compute the transformed equation and solve for $X(s) = \mathcal{L}\{x\}$:

$$\begin{aligned} (sX(s) - x_0) + kX(s) &= 0 \\ (s+k)X(s) &= x_0 \\ X(s) &= x_0 \left(\frac{1}{s+k} \right). \end{aligned}$$

We now untransform (i.e., take the inverse transform).

$$x(t) = x_0 \left(e^{-kt} \right).$$

II. First order with simple driver function

This situation usually requires some sort of partial-fractions decomposition. Consider the equation

$$x' + 5x = 2e^{-3t} \text{ with } x(0) = 1/2$$

(we divided through by the coefficient of x' as in the previous example). We transform (writing $X(s)$ as simply X):

$$\begin{aligned} (sX - 1/2) + 5X &= \frac{2}{s+3} \\ (s+5)X &= \frac{1}{2} + \frac{2}{s+3} = \frac{\frac{1}{2}s + \frac{7}{2}}{s+3} \\ X(s) &= \frac{\frac{1}{2}s + \frac{7}{2}}{(s+3)(s+5)}. \end{aligned}$$

In order to untransform this fraction, we write $\frac{\frac{1}{2}s + \frac{7}{2}}{(s+3)(s+5)}. = \frac{A}{s+3} + \frac{B}{s+5}$, so $\frac{1}{2}s + \frac{7}{2} = A(s+5) + B(s+3)$. Letting s first be -3 gives $A = 1$; letting s be -5 gives $B = -\frac{1}{2}$. Thus, $\frac{\frac{1}{2}s + \frac{7}{2}}{(s+3)(s+5)}. = \frac{1}{s+3} - \frac{1/2}{s+5}$ so

$$\begin{aligned} X(s) &= \frac{1}{s+3} - \frac{1/2}{s+5} \\ x(t) &= e^{-3t} - \tfrac{1}{2}e^{-5t}. \end{aligned}$$

III. Two useful formulas

When the driver function is a multiple of $\sin \omega t$ or $\cos \omega t$, the following formulas are helpful in simplifying the transform. (You do not have to memorize these.)

$$\begin{aligned} \frac{1}{(s+k)(s^2+\omega^2)} &= \frac{1}{k^2+\omega^2} \left[\frac{1}{s+k} - \frac{s}{s^2+\omega^2} + \frac{k}{s^2+\omega^2} \right] \\ \frac{s}{(s+k)(s^2+\omega^2)} &= \frac{-1}{k^2+\omega^2} \left[\frac{k}{s+k} - \frac{ks}{s^2+\omega^2} - \frac{\omega^2}{s^2+\omega^2} \right]. \end{aligned}$$

Exercises

1. Solve the initial-value problems

   (a) $x' - 4x = 0$, $x(0) = 0$.

   (b) $x' + 2x = 5$, $x(0) = -2$

   (c) $2x' + 7x = 3e^{-t}$, $x(0) = 4$

   (d) $2x' + x = 3t$, $x(0) = 3$ (Remember that a denominator with an s^2 requires $A/s + B/s^2$ in the partial fractions decomposition.)

2. The charge $Q(t)$ across a condenser of capacity C in series with a resistance R and a battery supplying constant voltage E satisfies the equation $RQ'(t) + Q(t)/C = E$. If $Q(0) = 0$, solve this equation.

3. Solve the initial-value problems

   (a) $2x' + 6x = 3\sin 2x$, $x(0) = 1$

   (b) $2x' + 6x = 3\cos 2x$, $x(0) = 1$.

4. To compare methods, use integrating factors to solve equation 1(b) above.

5. To compare methods, use integrating factors to solve equation 3(b) above.

6. In problem 2 above, replace the constant voltage with an alternating voltage $E(t) = E_0 \sin \omega t$.

7. Derive, or at least verify, either of the "useful formulas" above.