Bases and Dependence

Here are some general facts that we will use.

1. Homogeneous equations with more unknowns than equations always have a non-trivial (not all zero) solution.

   The reason is that the matix of coefficients will have more columns (unknowns or variables) than rows (equations). There can be only one leading or pivot variable per row, so there must be at least one variable which is not a pivot variable — i.e. there must be a free variable. Since this variable can be set to anything, there is a solution which is not all zeros.

2. If the first k columns of a matrix are linearly independent, then its row-reduced echelon form (rref) looks like:

$$\left(\begin{array}{|ccccc|c|ccc|}
\multicolumn{5}{c}{\overbrace{\hphantom{1\quad 0\quad 0\quad \cdots\quad 0}}^{k\text{ columns}}} & \multicolumn{1}{c}{} & \multicolumn{3}{c}{} \\
\hline
1&0&0&\cdots&0& &*&\cdots&*\\
0&1&0&\cdots&0& &*&\cdots&*\\
0&0&1&\cdots&0& &*&\cdots&*\\
\vdots&\vdots&\vdots&\vdots&\vdots& &\vdots&\vdots&\vdots\\
0&0&0&\cdots&1& &*&\cdots&*\\
\hline
0&0&0&\cdots&0& &*&\cdots&*\\
\vdots&\vdots&\vdots&\vdots&\vdots& &\vdots&\cdots&\vdots\\
0&0&0&\cdots&0& &*&\cdots&*\\
\hline
\end{array}\right) = \left(\begin{array}{|c|c|}
\hline
k\times k\text{ Identity}&\text{Other Stuff}\\
\hline
\text{All Zeros}&\text{Other Stuff}\\
\hline
\end{array}\right)$$

   To see this, let $\mathbf{V}$ be the matrix of just the first k columns (so it is $n\times k$), and let $\mathbf{R}$ be its row-reduced form. Then the equations $\mathbf{V}\cdot\mathbf{C}=\mathbf{0}$ and $\mathbf{R}\cdot\mathbf{C}=\mathbf{0}$ have the same solutions (because row-ops are reversible). Since the columns of $\mathbf{V}$ are independent, there is only one solution: $\mathbf{C}=\mathbf{0}$. But a row-reduced matrix with only one solution must have every variable a pivot variable; i.e. there must be exactly k pivots, one in each row. Thus, each of the first k rows must be a pivot row, and so the "top" part of $\mathbf{R}$ must be the $k\times k$ identity. Since every column of $\mathbf{R}$ is a pivot column, all the remaining "bottom" rows must be zero. Returning to the original matrix, this gives the left-hand side of the diagram above. Since we know nothing about the other column, that might be anything.

3. Suppose that $\mathbf{v}_1,\dots,\mathbf{v}_k$ are independent vectors (written as columns), and suppose $\mathbf{w}\in\mathrm{Span}(\mathbf{v}_1,\dots,\mathbf{v}_k)$. Then the augmented matrix $\mathbf{M}=\begin{pmatrix} | & | & \cdots & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_k & \mathbf{w} \\ | & | & \cdots & | & | \end{pmatrix}$ row reduces to a matrix that looks like

$$\left(\begin{array}{|c|c|}
\hline
k\times k\text{ Identity}&\text{Column of }k\text{ numbers}\\
\hline
\text{All Zeros}&\text{Column of zeros}\\
\hline
\end{array}\right).$$

   Here's why. Since $\mathbf{w}\in\mathrm{Span}(\mathbf{v}_1,\dots,\mathbf{v}_k)$, there is a solution to the k equations in k unknowns: $\lambda_1\mathbf{v}_1+\cdots+\lambda_k\mathbf{v}_k=\mathbf{w}$. The row-reduced matrix gives these solutions. From part 2 above, we know the first k columns must look like the $k\times k$ identity with zeros rows below. If there were any non-zero number r in the lower right column we would get the contradiction $\lambda_1\cdot 0+\cdots+\lambda_k\cdot 0=r$.

We now state our main result.

Theorem *Suppose $v_1, \ldots, v_k$ are linearly independent and $w_1, \ldots, w_{k+1}$ lie in* Span*($v_1, \ldots, v_k$); then w_1, $\ldots, w_{k+1}$ are linearly dependent.*

Proof. Let $\mathbf{A}$ be the matrix whose columns are the $2k+1$ vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{w}_1, \ldots, \mathbf{w}_{k+1}$:

$$\mathbf{A} = \begin{pmatrix} | & | & \cdots & | & | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_k & \mathbf{w}_1 & \cdots & \mathbf{w}_{k+1} \\ | & | & \cdots & | & | & | & | \end{pmatrix}.$$

We row-reduce $\mathbf{A}$ to get the row-reduced matrix $\mathbf{R}$, and we claim that $\mathbf{R}$ looks like:

$$\left(\begin{array}{cc} \overbrace{\begin{array}{|ccccc|}\hline 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \hline 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ \hline \end{array}}^{k \text{ columns}} & \begin{array}{|ccc|}\hline * & \cdots & * \\ * & \cdots & * \\ * & \cdots & * \\ \vdots & \vdots & \vdots \\ * & \cdots & * \\ \hline 0 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & 0 \\ \hline \end{array} \end{array}\right) = \left(\begin{array}{|c|c|}\hline k \times k \text{ Identity} & \text{Matrix } \mathbf{J} \\ \hline \text{All Zeros} & \text{All Zeros} \\ \hline \end{array}\right).$$

This follows directly from part 3 above, applied to each of the vectors $\mathbf{w} = \mathbf{w}_1, \ldots, \mathbf{w}_{k+1}$, since each is in Span($\mathbf{v}_1, \ldots, \mathbf{v}_k$). Now we note the key fact: the matrix $\mathbf{J}$ in the upper right corner has $k+1$ columns (one for each of the $\mathbf{w}_i$) and only k rows. By part 1 above, we can find a (column) vector $\mathbf{C}$ with entries $c_1, \ldots, c_{k+1}$, such that $\mathbf{J} \cdot \mathbf{C} = \mathbf{0}$, and $\mathbf{C}$ is not all zeros. ($\mathbf{C}$ gives a non-trivial dependence among the columns of $\mathbf{J}$.)

We now form a $k + (k+1)$ component column vector $\overline{\mathbf{C}}$ from $\mathbf{C}$ by adding k 0's to the beginning of **C.**

$$\overline{\mathbf{C}} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ c_1 \\ \vdots \\ c_{k+1} \end{pmatrix} \qquad \begin{array}{l} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} k \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} k+1 \end{array}.$$

We first note that $\mathbf{R}\cdot\overline{\mathbf{C}} = \mathbf{0}$. This is because the first k entries of $\overline{\mathbf{C}}$ are all zeros, so they multiply the first k column vectors of $\mathbf{R}$ into $\mathbf{0}$. The entries c_1 through c_{k+1} combine the upper, or "**J**" part, of the last $k+1$ columns of $\mathbf{R}$ into $\mathbf{0}$, and the remaining — lower — part of the columns is already zero; thus, $\mathbf{R}\cdot\overline{\mathbf{C}} = \mathbf{0}$. However, as already noted above, a matrix $\mathbf{A}$ and the matrix obtained from any row operations, say $\mathbf{R}$, always have the same solutions. So we conclude that $\mathbf{A}\cdot\overline{\mathbf{C}} = \mathbf{0}$ as well. This gives a non-trivial dependence among the columns of $\mathbf{A}$. But the first k entries of $\overline{\mathbf{C}}$ are zero, so this non-trivial dependence doesn't involve the first k columns of $\mathbf{A}$, which are the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$. Thus $\overline{\mathbf{C}}$ gives us a non-trivial dependence relation among the last $k+1$ column vectors of $\mathbf{A}$, which are the vectors $\mathbf{w}_1, \ldots, \mathbf{w}_{k+1}$. This completes the proof. ■

Corollary *Suppose $v_1, \ldots, v_k$ are linearly independent and $w_1, \ldots, w_n$ are also linearly independent and lie in* Span*($v_1, \ldots, v_k$); then $n \leq k$.*

Proof. If $n > k$ then the Theorem says that the w's would be dependent. ■

Corollary Any two bases for a vector space have the same number of elements.

Proof. If $\mathbf{v}_1, \ldots, \mathbf{v}_k$ and $\mathbf{w}_1, \ldots, \mathbf{w}_n$ are both bases, then each set of vectors is linearly independent and lies in the span of the other. By the previous corollary, $k \leq n$ and $n \leq k$; thus $k = n$. ■