DIFFERENTIAL EQUATIONS PRACTICE PROBLEMS

1. Find the solution of $y' + 2xy = x$, with $y(0) = -2$.

2. Find the general solution of $xy' = y - (y^2/x)$.

3. Suppose that the frog population $P(t)$ of a small lake satisfies the differential equation $\frac{dP}{dt} = kP(200 - P)$.

   (a) Find the equilibrium solutions. Sketch them and using the equation, sketch several solution curves, choosing some with initial points above and between the equilibrium solutions.

   (b) In the year 2000, its population was 100 and growing at the rate of 5 per year. Predict the lake's frog population in 2008. $\left(\text{Note} : \dfrac{1}{P(200-P)} = \dfrac{1/200}{P} + \dfrac{1/200}{(200-P)}.\right)$

4. Find the general solution of the differential equation $y'' - y' = e^x - 9x^2$.

5. A mass of 2 kg is attached to a spring with constant $k = 8$ Newtons/meter.

   (a) Find the natural frequency of this system.

   (b) If the motion is also subject to a damping force with $c = 4$ Newtons/(meter/sec), and the mass is initially pulled 1 meter beyond its equilibrium point and released (without initial velocity), find the motion, $x(t)$. (You may leave your answer in any form.)

6. Find *and sketch* the solution to the initial value problem $y'' + 4y = \delta(t - \pi)$, $y(0) = y'(0) = 0$.

7. Find the inverse Laplace transform:$F(s) = \frac{s^2+4}{s(s^2+1)}$

8. Let $A = \begin{pmatrix} 1 & 4 & 3 \\ 1 & 5 & 5 \\ 2 & 5 & 1 \end{pmatrix}$

   (a) Find A^{-1}, the inverse of A.

   (b) Use your answer above to solve $A\vec{x} = \vec{b}$ where $\vec{b} = (1, 0, -1)$.

9. The matrix

$$B = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$$

   has eigenvalues $\lambda = 4, -2$. Find a basis of eigenvectors.

10. The reduced row echelon form for the matrix A below has been computed by Matlab:

$$A = \begin{pmatrix} 2 & -4 & -1 & 2 \\ -3 & 6 & 1 & -5 \\ 5 & -10 & -4 & -1 \end{pmatrix} \qquad \text{rref}(A) = \begin{pmatrix} 1 & -2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

   Use this to find all solutions of

$$\begin{aligned} 2x_1 - 4x_2 - x_3 &= 2 \\ -3x_1 + 6x_2 + x_3 &= -5 \\ 5x_1 - 10x_2 - 4x_3 &= -1 \end{aligned}$$

   and express your answer in vector form.

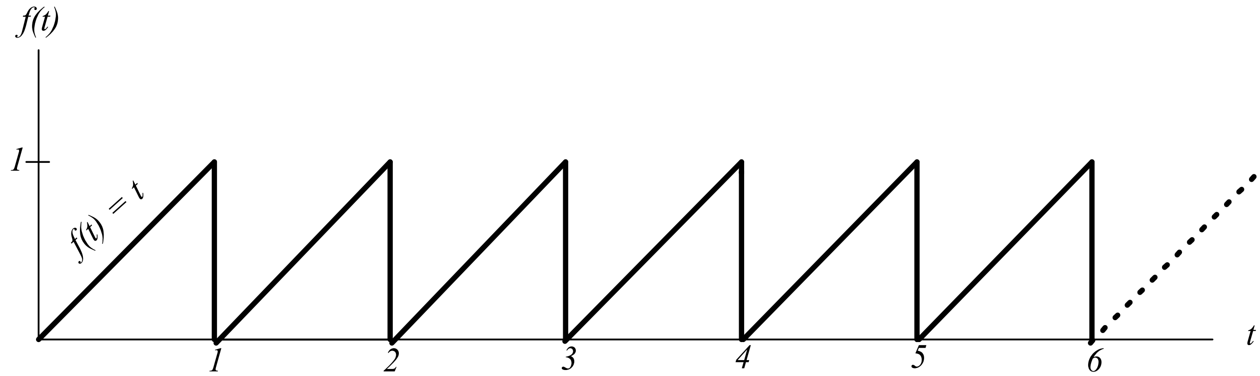
11. Let $\mathbf{v}_1 = (2, 1, 3)$, $\mathbf{v}_2 = (1, 5, 9)$, and $\mathbf{w} = (1, -1, -1)$. Is $\mathbf{w}$ in span$(\mathbf{v}_1, \mathbf{v}_2)$? Find a basis for span$(\mathbf{v}_1, \mathbf{v}_2, \mathbf{w})$. What is the *dimension* of span$(\mathbf{v}_1, \mathbf{v}_2, \mathbf{w})$?

12. Consider the following system of first-order differential equations:

$$\begin{aligned} x_1' &= 9x_1 + 5x_2 \qquad x_1(0) = 1 \\ x_2' &= -6x_1 - 2x_2 \qquad x_2(0) = 0 \end{aligned}$$

Use eigenvalues and eigenvectors to find the solution.

13. Here is a "sawtooth" function $f(t)$:



The first "tooth" is the function $f_1(t) = \begin{cases} t & \text{for } 0 \le t < 1 \\ 0 & \text{otherwise.} \end{cases}$.

(a) From the definition $\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty f(t)e^{-st}\,dt$, show that $F_1(s) = \frac{1-e^{-s}}{s^2} - \frac{e^{-s}}{s}$. (Use integration by parts; you only have to integrate from 0 to 1.)

(b) For a periodic function f of period p, $F(s) = \frac{1}{1-e^{-ps}} \int_0^p f(t)e^{-st}\,dt$. Use this and part (a) to show that, for the sawtooth:

$$F(s) = \frac{1}{s^2} - \frac{e^{-s}}{s(1-e^{-s})}.$$