1.

$$\begin{aligned} x_1' &= 2x_1 + 2x_2 \\ x_2' &= x_1 + 3x_2 \end{aligned}$$

(a) Write this system in matrix form (let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ etc.) (10 pts)

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{or } \mathbf{x}' = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} \cdot \mathbf{x}$$

(b) Use the method of *eigenvalues & eigenvectors* to find the general solution to this system. (35 pts)

Characteristic polynomial is $\begin{vmatrix} 2-x & 2 \\ 1 & 3-x \end{vmatrix} = (2-x)(3-x)-2 = x^2-5x+4 = (x-1)(x-4)$. Thus, the eigenvalues are $\lambda = 1, 4$ (15pts). We now find the corresponding eigenvectors: (15 pts)

$\lambda = 1$: $\begin{pmatrix} 2-1 & 2 \\ 1 & 3-1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$, so $a = -2b$, $b =$ anything. Let $b = -1$; then $\mathbf{v}_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

$\lambda = 4$: $\begin{pmatrix} 2-4 & 2 \\ 1 & 3-4 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$, so $a = b$, $b =$ anything. Let $b = 1$; then $\mathbf{v}_4 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

The general solution is then $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1\mathbf{v}_1 e^t + c_2\mathbf{v}_4 e^{4t} = \begin{pmatrix} 2c_1e^t + c_2e^{4t} \\ -c_1e^t + c_2e^{4t} \end{pmatrix}$. This can also be written $x_1(t) = 2c_1e^t + c_2e^{4t}$, and $x_2(t) = -c_1e^t + c_2e^{4t}$. (15 pts)

(c) Find the specific solution determined by the initial conditions $x_1(0) = 3$ and $x_2(0) = 1$. (20 pts)

Putting $t = 0$ in the expression for x_1 and x_2, and using the initial conditions, give the equations

$$\begin{aligned} 3 &= 2c_1 + c_2 \\ 1 &= -c_1 + c_2. \end{aligned}$$

These give $c_1 = 2/3$ and $c_2 = 5/3$, so

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{4}{3}e^t + \frac{5}{3}e^{4t} \\ -\frac{2}{3}e^t + \frac{5}{3}e^{4t} \end{pmatrix}.$$

2. Suppose the system $\mathbf{x}' = A\mathbf{x}$ has complex eigenvalue $-3+2i$, with eigenvector $\mathbf{v} = \begin{pmatrix} 1 \\ i \end{pmatrix}$. Find the general solution (in terms of trig functions). (35 pts)

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{-3t}(\cos 2t + i\sin 2t) = \begin{pmatrix} e^{-3t}(\cos 2t + i\sin 2t) \\ ie^{-3t}(\cos 2t + i\sin 2t) \end{pmatrix} = \underbrace{\begin{pmatrix} e^{-3t}\cos 2t \\ -e^{-3t}\sin 2t \end{pmatrix}}_{\text{real part}} + i\underbrace{\begin{pmatrix} e^{-3t}\sin 2t \\ e^{-3t}\cos 2t \end{pmatrix}}_{\text{imaginary part}}$$

Since the real and imaginary parts are always solutions, we get the general solution:

$$\begin{aligned}
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= c_1 \underbrace{\begin{pmatrix} e^{-3t}\cos 2t \\ -e^{-3t}\sin 2t \end{pmatrix}}_{\text{real part}} + c_2 \underbrace{\begin{pmatrix} e^{-3t}\sin 2t \\ e^{-3t}\cos 2t \end{pmatrix}}_{\text{imaginary part}} \\
x_1 &= c_1 e^{-3t}\cos 2t + c_2 e^{-3t}\sin 2t \\
x_2 &= -c_1 e^{-3t}\sin 2t + c_2 e^{-3t}\cos 2t
\end{aligned}$$