1. Use Laplace transforms (see table on back) to solve the initial-value problem

$$x'' + x' - 6x = 3t,\ x(0) = 1,\ x'(0) = 0$$

This transforms to $(s^2X - s) + (sX - 1) - 6X = \frac{3}{s^2}$ or:

$$X(s) = \frac{s^3 + s^2 + 3}{s^2(s+3)(s-2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3} + \frac{D}{s-2}.$$

Clearing fractions we get $s^3 + s^2 + 3 = As(s+3)(s-2) + B(s+3)(s-2) + Cs^2(s-2) + Ds^2(s+3)$. Letting s be 0, -3 and 2 yields $B = -\frac{1}{2}$, $C = \frac{1}{3}$ and $D = \frac{3}{4}$ respectively. Using $s = 1$ and these values gives $A = -\frac{1}{12}$. Thus

$$\begin{aligned} X(s) &= -\frac{1}{12}\cdot\frac{1}{s} - \frac{1}{2}\cdot\frac{1}{s^2} + \frac{1}{3}\cdot\left(\frac{1}{s+3}\right) + \frac{3}{4}\cdot\left(\frac{1}{s-2}\right) \\ x(t) &= -\frac{1}{12} - \frac{1}{2}t + \frac{1}{3}e^{-3t} + \frac{3}{4}e^{2t}. \end{aligned}$$

If the equation had been $x'' + x' - 6x = 3t^2$ with the same initial conditions, then we would have

$$\begin{aligned} X(s) &= \frac{s^3 + s^4 + 6}{s^3(s+3)(s-2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s+3} + \frac{E}{s-2} \\ s^3 + s^4 + 6 &= As^2(s+3)(s-2) + Bs(s+3)(s-2) + C(s+3)(s-2) + Ds^3(s-2) + Es^2(s+3). \end{aligned}$$

Letting $s = 0,\ 2,\ -3$ gives $C = -1$, $E = \frac{3}{4}$, $D = \frac{4}{9}$. Letting $s = 1$ and -1 gives simultaneous equations which yield $A = -\frac{5}{9}$, $B = -\frac{1}{12}$. Plugging these in and untransforming gives the solution

$$x(t) = -\frac{5}{9} - \frac{t}{12} - \frac{1}{2}t^2 + \frac{4}{9}e^{-3t} + \frac{3}{4}e^{2t}.$$

2. Use Laplace transforms to solve the initial-value problem

$$x'' + 9x = 2\sin(7t),\ x(0) = x'(0) = 0.$$

(Hint: In doing the partial fractions, you just need constants A and B in the numerators since there won't be any odd powers of s.)

Transforming, we get $s^2X + 9X = \frac{2\cdot 7}{s^2+49}$. This gives

$$\begin{aligned} X(s) &= \frac{14}{(s^2+9)(s^2+49)} = \frac{A}{(s^2+9)} + \frac{B}{(s^2+49)} \\ 14 &= A\cdot(s^2+49) + B\cdot(s^2+9). \end{aligned}$$

Letting $s = 0$ and $s = 1$ gives the simultaneous equations $14 = 49A + 9B$ and $14 = 50A + 10B$, whose solutions are $A = \frac{7}{20}$ and $B = -\frac{7}{20}$. Thus

$$\begin{aligned} X(s) &= \frac{7}{20}\frac{1}{s^2+9} - \frac{7}{20}\frac{1}{s^2+49} = \frac{7}{20}\frac{1}{3}\frac{3}{s^2+9} - \frac{7}{20}\frac{1}{7}\frac{7}{s^2+49}. \\ x(t) &= \frac{7}{60}\sin(3t) - \frac{1}{20}\sin(7t). \end{aligned}$$