

1. Calculate the following.

(a) $\mathcal{L}\{e^{-3t}t^2\}$

The basic function here is t^2 whose transform is $\frac{2}{s^3}$; however, the e^{-3t} multiplier creates a translation (see table), resulting in the answer $\frac{2}{(s+3)^3}$.

(b) $\mathcal{L}^{-1}\left\{\frac{7}{s^2+8s+41}\right\}$

Since the polynomial doesn't factor, we complete the square, giving $\frac{7}{(s+4)^2+5^2}$. The basic function here is $\sin 5t$, but the $s+4$ means there was a translation; we also need a 5 in the numerator:

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{7}{s^2+8s+41}\right\} &= \mathcal{L}^{-1}\left\{\frac{7}{(s+4)^2+5^2}\right\} = \frac{7}{5}\mathcal{L}^{-1}\left\{\frac{5}{(s+4)^2+5^2}\right\} \\ &= \frac{7}{5}e^{-4t}\sin 5t.\end{aligned}$$

(c) $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\}$

Forget the e^{-2s} till the end. The inverse-transform of $\frac{1}{s^2}$ is t our "basic" function here. Multiplying now by the e^{-2s} causes a translation (see formulas), resulting in the answer $u(t-2) \cdot (t-2)$.

(d) $\mathcal{L}^{-1}\left\{\frac{e^{-7s}}{s+5}\right\}$

Once again, leaving the e^{-7s} for last, we see that the basic function here is $\frac{1}{s+5}$ whose inverse transform is e^{-5t} . Returning to the e^{-7s} we see that we need a translation of the basic function by 7, resulting in the answer $u(t-7)e^{-5(t-7)}$.

2. Let $f(t) = \begin{cases} \sin 2t & \text{for } 0 \leq t < \pi \\ 0 & \text{for } t \geq \pi \end{cases}$.

(a) Explain why $f(t) = [1 - u(t - \pi)] \sin 2(t - \pi)$.

When $0 \leq t < \pi$, $u(t - \pi) = 0$, so $[1 - u(t - \pi)] \sin 2(t - \pi) = \sin 2(t - \pi) = \sin(2t - 2\pi) = \sin 2t$.

When $t \geq \pi$, $u(t - \pi) = 1$, so $[1 - u(t - \pi)] \sin 2(t - \pi) = [1 - 1] \sin 2(t - \pi) = 0$. Thus, the function agrees with $f(t)$.

(b) Using part a, calculate $F(s) = \mathcal{L}\{f(t)\}$

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \mathcal{L}\{[1 - u(t - \pi)] \sin 2(t - \pi)\} \\ &= \mathcal{L}\{\sin 2t\} - \mathcal{L}\{u(t - \pi) \sin 2(t - \pi)\} \\ &= \frac{2}{s^2 + 4} - e^{-\pi s} \frac{2}{s^2 + 4} \\ &= \frac{2}{s^2 + 4} (1 - e^{-\pi s}).\end{aligned}$$

(c) Let $f_{3\pi} = \begin{cases} \sin 2t & \text{for } 3\pi \leq t < 4\pi \\ 0 & \text{for } t < 3\pi \text{ and } t \geq 4\pi \end{cases}$. Calculate $\mathcal{L}\{f_{3\pi}(t)\}$

The easiest way to do this is to note that $f_{3\pi}(t) = u(t - 3\pi)f(t - 3\pi)$ so

$$\begin{aligned} \mathcal{L}\{f_{3\pi}\} &= e^{-3\pi s} \cdot \frac{2}{s^2 + 4} (1 - e^{-\pi s}) \\ &= \frac{2}{s^2 + 4} (e^{-3\pi s} - e^{-4\pi s}). \end{aligned}$$

Another approach is to notice directly that $f_{3\pi}(t) = [u(t - 3\pi) - u(t - 4\pi)]f(t)$, and also $f(t) = f(t - 3\pi) = f(t - 4\pi)$.

Laplace Transforms

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} f(t)e^{-st} dt$$

$f(t)$	$F(s)$	Domain of F
1	$\frac{1}{s}$	$s > 0$
t	$\frac{1}{s^2}$	$s > 0$
t^n	$\frac{n!}{s^{\alpha+1}}$	$s > 0$
$e^{\alpha t}$	$\frac{1}{s - \alpha}$	$s > \alpha$
$\sin(kt)$	$\frac{k}{s^2 + k^2}$	$s > 0$
$\cos(kt)$	$\frac{s}{s^2 + k^2}$	$s > 0$
$u(t - a)$	$\frac{e^{-as}}{s}$	$s > 0$
$\delta_a(t)$	e^{-as}	$s, a \geq 0$

Other useful facts:

$$\begin{cases} \mathcal{L}\{x'(t)\} = sX(s) - x(0) \\ \mathcal{L}\{x''(t)\} = s^2X(s) - sx(0) - x'(0) \\ \mathcal{L}\{e^{at}f(t)\} = F(s - a) \\ \mathcal{L}^{-1}\{F(s - a)\} = e^{at}f(t) \\ \mathcal{L}\{u(t - a)f(t - a)\} = e^{-as}F(s) \\ \mathcal{L}^{-1}\{e^{-as}F(s)\} = u(t - a)f(t - a) \end{cases}$$