1. Let $\mathbf{v}_1 = (1,2,3)$, $\mathbf{v}_2 = (0,1,0)$, $\mathbf{v}_3 = (2,1,6)$.

   (a) Find out if these three vectors are dependent. If they are, write down a non-trivial dependence relation.

   Make these into the columns of a matrix A. A dependence relation is a solution to $A\mathbf{X} = \mathbf{0}$, so we row reduce to see if there is a non-trivial solution:

$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 3 & 0 & 6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}, \ z = s,\ y = 3s,\ x = -2s.$$

   Letting $s = 1$ gives a non-trivial solution: $-2\mathbf{v}_1 + 3\mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$, or $\mathbf{v}_3 = 2\mathbf{v}_1 - 3\mathbf{v}_2$.

   (b) Determine whether $\mathbf{w} = (1,1,1)$ lies in the *span* of $\mathbf{v}_1$ and $\mathbf{v}_2$.

   We want to solve $x\mathbf{v}_1 + y\mathbf{v}_2 = \mathbf{w}$, so we make the *augmented matrix* with columns $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{w}$, and row reduce:

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 3 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -2 \end{array}\right]$$

   This translates into $x = 1$, $y = -1$, and $0 = -2$. This this last equation is impossible, there is *no solution.*

2. Below is the result of row-reduction of the matrix $\mathbf{A}$. Use $\mathbf{E}$ in answering the questions below.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & 0 & -2 & 0 & 0 \\ -5 & -10 & 1 & -1 & 11 & -2 & -11 \\ 12 & 24 & -2 & 3 & -24 & 6 & 33 \\ 6 & 12 & -1 & 1 & -13 & 3 & 15 \\ 2 & 4 & 0 & 1 & -2 & 2 & 11 \end{bmatrix} \quad \text{row reduces to} \quad \mathbf{E} = \begin{bmatrix} 1 & 2 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   Denote by $\mathbf{v}_1, \dots, \mathbf{v}_7$ the columns of $\mathbf{A}$

   (a) First consider $\mathbf{C} = \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_7)$. Find a subset of the column vectors of $\mathbf{A}$ which also span $\mathbf{C}$ but which are *independent.*

   The four pivot columns of $\mathbf{E}$ are numbers $1, 3, 4$, and 6, so the corresponding column vectors of $\mathbf{A}$, $\mathbf{v}_1$, $\mathbf{v}_3$, $\mathbf{v}_4$, and $\mathbf{v}_6$, are independent.

   (b) Consider the system of equations $\mathbf{Ax} = \mathbf{0}$ where $\mathbf{x}$ is a column of unknowns. Three of these unknowns are *free* variables: which? How many pivot variables will there be?

   Columns $2, 5$, and 7 are not pivot columns, hence are free; thus, there are 4 pivot variables (as in part b).

   (c) If the free variables are set equal to r, s, and t, then write the general solution to $\mathbf{Ax} = \mathbf{0}$ in the form $\mathbf{x} = r \cdot \mathbf{w}_1 + s \cdot \mathbf{w}_2 + t \cdot \mathbf{w}_3$ the way we usually do (the $\mathbf{w}$'s are column vectors).

   If we set $x_7 = r$, $x_5 = s$ and $x_2 = t$, then we read off:

$$\begin{array}{rcl} x_1 &=& -2t + 2s \\ x_2 &=& t \\ x_3 &=& -3s \\ x_4 &=& -2s - 3r \\ x_5 &=& s \\ x_6 &=& -4r \\ x_7 &=& r \end{array}, \text{ so } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = r \begin{bmatrix} 0 \\ 0 \\ 0 \\ -3 \\ 0 \\ -4 \\ 1 \end{bmatrix} + s \begin{bmatrix} 2 \\ 0 \\ -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

(d) $\mathbf{C}$, the space spanned by the column vectors, is called the column space of $\mathbf{A}$. Its dimension (size of a basis) is called the *rank* of $\mathbf{A}$. What is the rank of $\mathbf{A}$ in this case?

From part (a) we see that $\text{rank}(\mathbf{A}) = 4$.

(e) The dimension of the solution space is called the *nullity* of $\mathbf{A}$. What is the nullity of $\mathbf{A}$ in this case?

From part (c) we see that $\text{nullity}(\mathbf{A}) = 3$.

(f) There is a general fact about the sum: $\text{rank}(\mathbf{A}) + \text{nullity}(\mathbf{A})$; what do you think it is?

For *any matrix* $\mathbf{A}$, $\text{rank}(\mathbf{A}) + \text{nullity}(\mathbf{A}) =$ number of columns of $\mathbf{A}$.

This is because the rank is the number of pivot columns and the nullity is the number of free columns.