1. Use the *wronskian* to show that the functions e^x and $\sin x$ are linearly independent.

   $W(e^x, \sin x) = \begin{vmatrix} e^x & \sin x \\ e^x & \cos x \end{vmatrix} = e^x \cos x - e^x \sin x = e^x(\cos x - \sin x)$. Since e^x is never zero, this can only be zero when $\cos x = \sin x$. Since this can't happen for all x, we see that the wronskian can't vanish identically, so e^x and $\sin x$ can't be dependent.

2. Find the *general solution* to the differential equation $25x'' + 10x' + x = 0$ (t is the independent variable).

   This has the complementary equation $25D^2 + 10D + 1 = 0$, or $(5D + 1)^2 = 0$. There is only the repeated root $D = -\frac{1}{5}$. Thus, the general solution is

   $$x(t) = c_1 e^{-t/5} + c_2 t e^{-t/5} = (c_1 + c_2 t)e^{-t/5}.$$

3. Find the general solution to the differential equations $x'' + 2x' - 15x = 0$.

   We have for the complementary equation $(D + 5)(D - 3) = 0$, with distinct roots $D = -5$ and $D = 3$. Thus, the general solution is

   $$x(t) = c_1 e^{-5t} + c_2 e^{3t}.$$

   (a) Find a solution satisfying the initial conditions $x(0) = 3$ and $x'(0) = -1$.
   $x'(t) = -5c_1 e^{-5t} + 3c_2 e^{3t}$. Plugging in $t = 0$ gives the equations

   $$\begin{aligned} 3 &= c_1 + c_2 \\ -1 &= -5c_1 + 3c_2 \end{aligned}$$

   with solutions $c_1 = 5/4$ and $c_2 = 7/4$. Thus, the specific solution is$x(t) = (5/4)e^{-5t} + (7/4)e^{3t}$.

   (b) Is this the *only* solution satisfying these initial conditions? How do you know?
   This is the only solution because the *Existence and Uniqueness Theorem* for linear differential equations tell us there is one and only one solution satisfying given initial conditions.