1. Consider the differential equation $x'' + 100x = \cos(\omega t)$.

   (a) For what value of ω will "resonance" take place?

   The solution to $D^2 + 100 = 0$ is $D = \pm 10i$ so the *homogeneous solution* is $x_h = c_1 \cos 10t + c_2 \sin 10t$. Thus, resonance takes place when $\omega = 10$.

   (b) In using the method of undetermined coefficients, what form will your "guess" for a *particular solution* take? (Do not actually find it.)

   Without resonance we would guess $x_p = A\cos 10t + B\sin 10t$; however, in the presence of resonance, we must adjust this to:

$$x_p(t) = t(A\cos 10t + B\sin 10t)$$

2. A *particular* solution to $2x'' + 2x' + x = 65\sin(2t)$ is $x_p = -4\cos(2t) - 7\sin(2t)$.

   (a) What is the *general solution*?

   The general solution to a linear equation is the general solution to the associated homogeneous equation *plus* any solution to the original. Solving $2D^2 + 2D + 1 = 0$ gives $D = \dfrac{-2 \pm \sqrt{4-8}}{4} = -\frac{1}{2} \pm \frac{1}{2}i$, so $x_h(t) = e^{-\frac{1}{2}t}(c_2 \cos \frac{1}{2}t + c_2 \sin \frac{1}{2}t)$ and the general solution is:

$$x_g(t) = e^{-\frac{1}{2}t}(c_2 \cos \tfrac{1}{2}t + c_2 \sin \tfrac{1}{2}t) - 4\cos 10t - 7\sin 10t.$$

   (b) In your general solution, label the *transcient* and *steady state* parts. Why are they given these names?

   $x_g(t) = \underbrace{e^{-\frac{1}{2}t}(c_2 \cos \frac{1}{2}t + c_2 \sin \frac{1}{2}t)}_{\text{transcient}} \overbrace{-4\cos 10t - 7\sin 10t}^{\text{steady state}}$. Transcient means "short-lived": it is called that since, when there is damping, there is always a negative exponential coefficient $e^{-\alpha t}$ which makes this term approach 0 at time increases. Thus, after some time elapses, the entire solution is very close to the remaining piece, which is therefore called steady state; it is often pure harmonic.

3. Find the *eigenvalues* of the following matrix — *do not bother with the eigenvectors.*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -6 & 8 & 2 \\ 12 & -15 & -3 \end{bmatrix}$$

$$\begin{aligned}
\chi_A(\lambda) &= \det(A - xI) = \det \begin{bmatrix} 1-\lambda & 0 & 0 \\ -6 & 8-\lambda & 2 \\ 12 & -15 & -3-\lambda \end{bmatrix} \\
&= (1-\lambda) \begin{vmatrix} 8-\lambda & 2 \\ -15 & -3-\lambda \end{vmatrix} = (1-\lambda)\left[(8-\lambda)(-3-\lambda) + 30\right] \\
&= (1-\lambda)(\lambda^2 - 5\lambda + 6) = (1-\lambda)(\lambda-2)(\lambda-3).
\end{aligned}$$

Thus, the eigenvalues are $\lambda = 1, 2, 3$.

4. One eigenvalue of the matrix $\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$ is 5. Find an *eigenvector* associated with this eigenvalue.

We must find the solutions of the matrix equation $\begin{bmatrix} 4-5 & 1 \\ 3 & 2-5 \end{bmatrix} \mathbf{v} = \mathbf{0}$. We row reduce:

$$\begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}.$$

Thus, $x_2 = s$ and $x_1 - x_2 = 0$, so (letting $s = 1$): $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.