The continuity of $xf(x)$ and an example

Suppose that $f(x)$ is uniformly continuous on $[A,B]$ and bounded by K; that is, $f(x)$ is UC and $|f(x)| \le K$ for all $x \in [A,B]$. Let $M = \max(|A|, |B|)$.

$$\begin{aligned} |yf(y) - xf(x)| &= |yf(y) - yf(x) + yf(x) - xf(x)| \\ &\le |yf(y) - yf(x)| + |yf(x) - xf(x)| \\ &= |y| \cdot |f(y) - f(x)| + |f(x)| \cdot |y - x| \\ &\le M \cdot |f(y) - f(x)| + K \cdot |y - x| \end{aligned}$$

(Here we have used first the triangle inequality, then the fact that $|uv| = |u| \cdot |v|$, and finally that $|y| \le M$ and $|f(x)| \le K$.)

Now suppose that we are given $\epsilon > 0$; we must make the above expression $< \epsilon$ by making $|y - x|$ sufficiently small ($<$ some δ). We will make each of the two added pieces $< \epsilon/2$.

Since f is UC and $\dfrac{\epsilon/2}{M} > 0$, we can find a *modulus of continuity* $\delta_f\left(\dfrac{\epsilon}{2M}\right)$ such that $|f(y) - f(x)| < \left(\dfrac{\epsilon}{2M}\right)$ when $|y - x| < \delta_f\left(\dfrac{\epsilon}{2M}\right)$ (all on $[A,B]$ of course). Also, $K \cdot |y - x|$ will be less than $\epsilon/2$ when $|y - x| < \dfrac{\epsilon}{2K}$. Since we want *both* conditions to hold, we let

$$\delta = \min\left(\delta_f\left(\frac{\epsilon}{2M}\right), \frac{\epsilon}{2K}\right).$$

Thus, when $|y - x| < \delta$ we'll have $|y - x| < \delta_f\left(\dfrac{\epsilon}{2M}\right)$ *and* $|y - x| < \dfrac{\epsilon}{2K}$, so we get

$$\begin{aligned} |yf(y) - xf(x)| &\le M \cdot |f(y) - f(x)| + K \cdot |y - x| \\ &\le M \cdot \left(\frac{\epsilon}{2M}\right) + K \cdot \left(\frac{\epsilon}{2K}\right) \\ &= \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

Done.

Now let's work out an actual example: $f(x) = 2x^3 - x^2 + 4$ on the interval $[-1,3]$; we suppose that we are given $\epsilon = 1/000$. We will need a modulus of continuity $\delta_f(\epsilon)$ for f on $[1,3]$ as well as a bound K. Let's do K first:

$$\begin{aligned} |f(x)| &= \left|2x^3 - x^2 + 4\right| \\ &\le \left|2x^3\right| + \left|x^2\right| + |4| = 2|x|^3 + |x|^2 + 4 \\ &\le 2(3)^3 + (3)^2 + 4 = 67; \end{aligned}$$

thus, $|f(x)| \le 67$ so we can take $K = 67$. (The actual "best" bound is 49.) Now let's find a modulus of continuity:

$$\begin{aligned}
|f(y)-f(x)| &= |\left(2y^3-y^2+4\right)-\left(2x^3-x^2+4\right)| \\
&= \left|2(y^3-x^3)-\left(y^2-x^2\right)\right| = |y-x|\cdot\left|(2y^2+2yx+2x^2)-(y+x)\right| \\
&\leq |y-x|\cdot\left(2\,|y|^2+2\,|y|\,|x|+2\,|x|^2+|y|+|x|\right) \\
&\leq |y-x|\cdot(2(3^2)+2(3)(3)+2(3^2)+3+3) \\
&= 60\cdot|y-x|\,.
\end{aligned}$$

To make $|f(y)-f(x)|<\epsilon$ it suffices to make $|y-x|<\epsilon/60$, so we can take $\delta_f(\epsilon)=\epsilon/60$. Finally, note that $M=\max(A,B)=\max(-1,3)=3$. Now we compute:

$$\begin{aligned}
\delta_f\left(\frac{\epsilon}{2M}\right) &= \left(\frac{\epsilon}{2M}\right)/60 = (1/1000)(1/2)(1/3)(1/60) = 1/360,000 \\
\left(\frac{\epsilon}{2K}\right) &= (1/000)(1/2)(1/67) = 1/134,000; \\
\delta &= \min\left(\delta_f\left(\frac{\epsilon}{2M}\right),\frac{\epsilon}{2K}\right) = 1/360,000.
\end{aligned}$$

Thus, if $|y-x|<1/360,000$ on $[-1,3]$, then $|yf(x)-xf(x)|=|(2y^3-y^2+4)-(2x^3-x^2+4)|$ $<1/1000$.