Visibles Revisited

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Introduction

Many notions in elementary and advanced number theory can be phrased in terms of lattice points — those points having integer coordinates (for example, see [**8**]). In this article we focus on one that has a particularly simple formulation. Suppose that an infinitely thin but opaque tree is planted at each lattice point. Points that are not screened from the origin by trees are called *visible points* or simply *visibles.* We can ask several basic questions about these points.

1. Is there a simple test for visibility?
2. Is there an algorithm for enumerating visibles?
3. What percentage of lattice points are visible?

Finding an answer to the first question provided a starting point for a recent first-year "Math Discovery" course at Northeastern University, and soon led to the others. Giving complete answers draws in results from many branches of mathematics: number theory, of course, but also lattice geometry and linear algebra, a few combinatorial arguments, and a non-trivial limit computation illustrating the methods of real analysis. The Riemann zeta function makes an appearance, and Euler's product formula ties it in with probability theory. Unfortunately the time margin for the course was too small to contain all these, so a lot had to be merely hinted at. Our goal in this paper, then, is to put these pieces together properly, while providing the necessary background material to make the exposition self-contained. The results we present are not new, but collecting them in one place shows, in a striking fashion, the interconnectedness of mathematics. In addition, visible points have become interesting to physicists because of their connections with crystal lattices and diffraction spectra [**1**].

Enumerating visibles

In the plane $\mathbb{R}^2$ we call the points with integer coordinates *lattice points.* A lattice point P is called *visible* if the line segment joining P to the origin contains no other lattice points; by convention, the origin itself is not visible. (For a selection of references on visible lattice points, see [**12**].) It is not difficult to show that a lattice point (m, n) is visible if and only if

$\gcd(m, n) = 1$, which answers the first question posed above. Since lattice points are essentially the same in each quadrant, we will assume $m, n \geq 0$. With the simple criterion of coprimeness, it is now easy to draw and count visibles that lie in some bounded region, say in the square $0 \leq x, y \leq N$ for small values of N. Figure 1 shows the visibles for $N = 4$, and Figure 2 for $N = 8$.

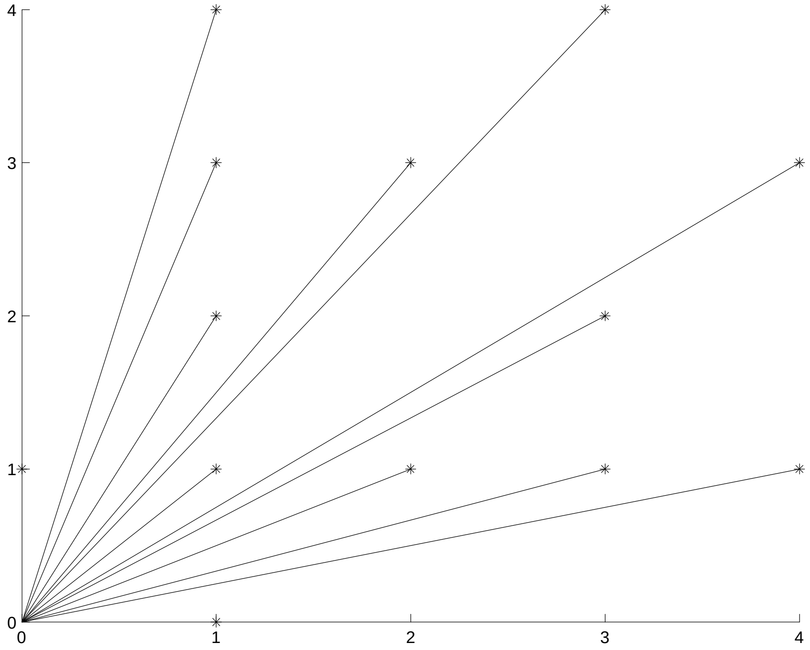

Figure 1: Visibles for N = 4

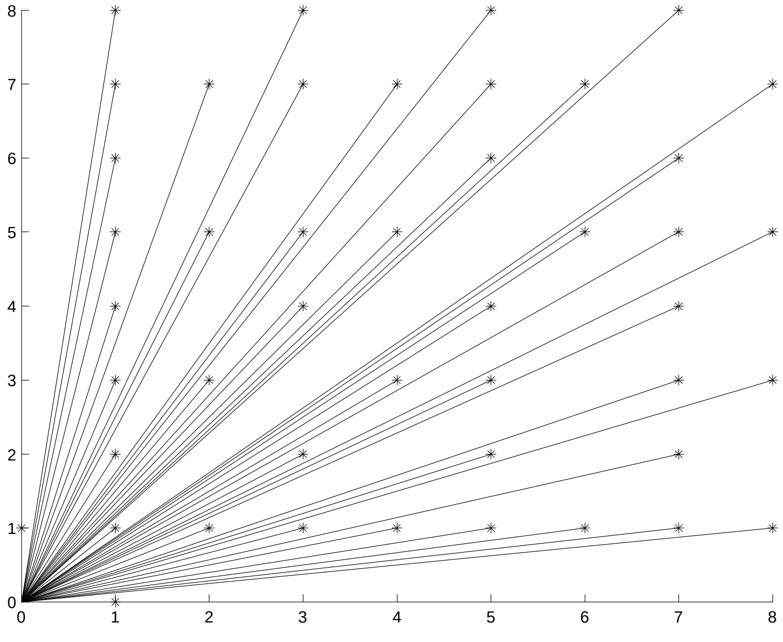

Figure 2: Visibles for N = 8

Clearly, the set of all visibles is symmetric about the coordinate axes, as well as about the diagonals $x = y$ and $x = -y$. Hence, in enumerating the visibles with $|m|$, $|n| \leq N$, we concentrate on those lying in the triangle $\mathcal{T}_N = \{(x,y) : 0 \le y \le x \le N\}$. We define the *visibility sequence* $\mathcal{V}_N$ to be the sequence of all visibles (m,n) for which $0 \le n \le m \le N$, ordered by the size of the slope n/m.

An easy inspection produces the lists $\mathcal{V}_N$ for $N \le 4$.

$$\begin{aligned}
\mathcal{V}_1 &= \{(1,0),(1,1)\} \\
\mathcal{V}_2 &= \Big\{(1,0),\underline{(2,1)},(1,1)\Big\} \\
\mathcal{V}_3 &= \Big\{(1,0),\underline{(3,1)},(2,1),\underline{(3,2)},(1,1)\Big\} \\
\mathcal{V}_4 &= \Big\{(1,0),\underline{(4,1)},(3,1),(2,1),(3,2),\underline{(4,3)},(1,1)\Big\}
\end{aligned}$$

We have underlined the "new" points obtained by expanding the size of the triangle. Note that each new point is the vector sum of the two older adjacent points. This holds in general, and is the basis of the following recursive enumeration algorithm.

The Visible Enumeration Theorem. For $N \ge 1$, the list $\mathcal{V}_{N+1}$ is obtained from $\mathcal{V}_N$ by inserting the point $(m+m', n+n')$ between any two consecutive visibles (m,n) and (m',n') in $\mathcal{V}_N$ for which $m + m' = N + 1$.

There are many algebraic proofs of this fact (e.g. [**9**], Theorem 2, p. 10), often couched in the language of Farey fractions (we discuss these in the next section). Instead, we turn to Pick's Theorem for a geometric argument.

Pick's Theorem. *Let P be a simple polygon with vertices at lattice points. Then the area of P is given by $I(P) + \frac{1}{2}B(P) - 1$, where $I(P)$ is the number of lattice points inside P and $B(P)$ is the number of lattice points on the boundary of P.*

(A "simple" polygon is one whose boundary is a closed and non-self-intersecting polygonal arc.)

We will not prove Pick's Theorem here (see [**2**], [**5**] and [**11**] for proofs). For our purposes, we only need the following special case.

Proposition 1. *A parallelogram in which the vertices are the only lattice points has area 1.*

Proof. We use some elementary linear algebra to prove this. First, we can translate the parallelogram by subtracting one of its vertices, so we can assume its vertices are $O = (0,0)$, $A = (m,n)$, B, and $A+B$. This does not introduce any new lattice points, so A is visible; hence $\gcd(m,n) = 1$ and we can find integers p and q with $mp + nq = 1$. Now consider the matrix $M = \begin{pmatrix} p & q \\ -n & m \end{pmatrix}$. It has integer entries and so does its inverse: $M^{-1} = \begin{pmatrix} m & -q \\ n & p \end{pmatrix}$. Thus, M establishes a bijection between the lattice points of our parallelogram and those of its image; since $\det(M) = 1$ this bijection also preserves area. Thus, in proving our statement, we can replace our parallelogram with its image under M; since $M \cdot A = (1,0)$, the new vertices are: $(0,0)$, $(1,0)$, (u,v), and $(u+1,v)$. Without loss of generality we can assume that $v \ge 1$. However, unless $v = 1$ the intersection of our parallelogram with the horizontal line $y = 1$ is a line segment through $(x, 1)$ and $(x + 1, 1)$ for some *non-integer* x, so it would contain a non-vertex lattice point. Thus, $v = 1$ and our parallelogram has area 1, as claimed. ■

Turning to the Visible Enumeration Theorem, we deduce it from the following statement.

Proposition 2. *Let O be the origin, let A and B be consecutive visibles in V_N, and let $S = A + B$.*

(a) The only lattice points of the parallelogram $OASB$ are its vertices.

(b) If M is the x-coordinate of S, then $M > N$, and S is the only visible between A and B in $\mathcal{V}_M$.

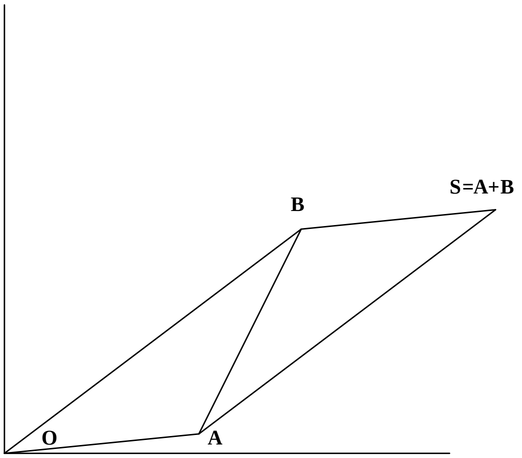

Figure 3. The parallelogram determined by consecutive visibles.

Proof. Since A and B are consecutive visibles in $\mathcal{V}_N$, $\Delta\ OAB$ has no lattice points other than vertices. The same is true for $\Delta\ ASB$, since it is obtained from $\Delta\ OAB$ by the transformation $Q \mapsto S - Q$. This proves the first statement. As a consequence, we see that S is a visible. It is now clear that $M > N$, since A and B are consecutive visibles in $\mathcal{V}_N$, and so S does not belong to $\mathcal{V}_N$.

To finish the proof, it remains to show that A and S (and also S and B) are consecutive visibles in $\mathcal{V}_M$. Suppose this is not so, and let P be the next visible after A in $\mathcal{V}_M$.

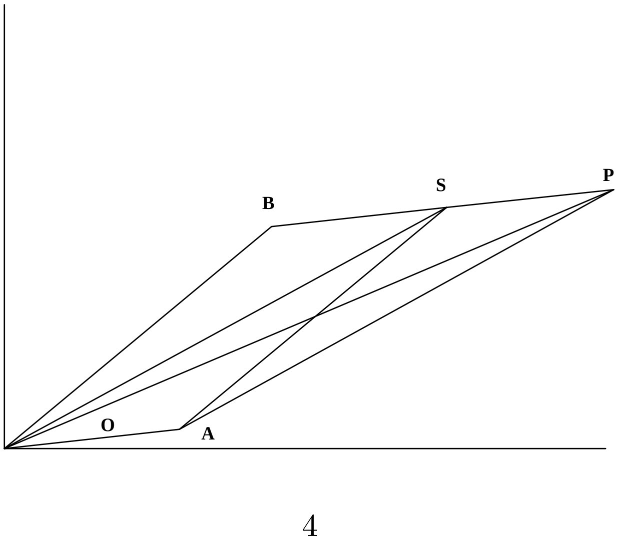

Figure 4. A, $S = A + B$, and B are consecutive visibles.

In view of Proposition 1, triangles $\Delta\ OAP$ and $\Delta\ OAS$ have lattice points only at vertices, so have area 1/2. Since the slope of OP lies between the slopes of OA and OS (so in particular, S and P are on one side of OA), we conclude that the vector SP is parallel to OA, and it points away from the origin. But then the x-coordinate of P must be greater than M, which makes the inclusion $P \in \mathcal{V}_M$ impossible. A similar argument shows that S and B are also consecutive visibles in $\mathcal{V}_M$, which completes the proof. ■

To prove the Enumeration Theorem, we apply Proposition 2(b), to $A = (m, n)$ and $B = (m', n')$, assuming that $m + m' = N + 1$. We deduce that $S = (m + m', n + n')$ is the only visible between A and B belonging to $\mathcal{V}_{N+1}$. If $m + m' > N + 1$, then Proposition 2(b) tells us that $\mathcal{V}_{N+1}$ has no points between A and B; this establishes the theorem.

Application to Farey Sequences

For a positive integer N, the *Farey sequence* F_N *of order* N is defined to be the list of all proper fractions, in lowest terms and in increasing order, whose denominators do not exceed N.

For example,

$$\begin{aligned}
\mathcal{F}_1 &= \frac{0}{1}, \frac{1}{1}\\
\mathcal{F}_2 &= \frac{0}{1}, \frac{1}{2}, \frac{1}{1}\\
\mathcal{F}_3 &= \frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\\
\mathcal{F}_4 &= \frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1},\\
&\vdots\\
\mathcal{F}_8 &= \frac{0}{1}, \frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{3}{8}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{5}{8}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{6}{7}, \frac{7}{8}, \frac{1}{1}
\end{aligned}$$

There is a natural order-preserving bijective correspondence between the Farey sequence $\mathcal{F}_N$ and the list $\mathcal{V}_N$ studied above; it is given by $p/q \leftrightarrow (q, p)$. As a consequence of what we have proved about visible points, we get the following basic facts about Farey sequences.

Proposition 3. *Let* $\frac{p}{q}$ *and* $\frac{p'}{q'}$ *be consecutive terms in* F_N*; then*

(a) $p'q - q'p = 1$;

(b) *the sum* $M = q + q'$ *is greater than* N*, and the only intermediate term between* $\frac{p}{q}$ *and* $\frac{p'}{q'}$ *in* F_M *is their mediant* $\frac{p+p'}{q+q'}$.

Part (a) is a consequence of Proposition 1, together with (a) of Proposition 2 and the determinant expression for the area of a parallelogram. Part (b) follows from Proposition 2(b). ■

Density of visibles

We now turn our attention to the last problem we posed in the Introduction: determine the *density* of visibles. Of course, we must say exactly what we mean by this, so we define ρ, the density of visibles, by:

$$\rho = \lim_{N\to\infty} \frac{|\mathcal{V}_N|}{\frac{1}{2}(N+1)(N+2)} \ , \tag{1}$$

where $\frac{1}{2}(N+1)(N+2)$ is the number of lattice points (m, n) with $0 \le n \le m \le N$.

Here is the answer.

The Density Theorem. *The density of visibles is* $\rho = \dfrac{6}{\pi^2}$.

As a corollary, we obtain the rate of growth for the Farey sequences.

Corollary. As $N \to \infty$, the length of the Farey sequence $\mathcal{F}_N$ grows as

$$\left(\frac{3}{\pi^2}\right) N^2 \approx (0.30396355)N^2.$$

In order to prove the Density Theorem we need to obtain a useful formula for the numerator $|\mathcal{V}_N|$ in the expression for the density. One approach is to group together the visibles $(m, n) \in \mathcal{V}_N$ according to the value of m. Counting them in this way we see that $|\mathcal{V}_N|$ is equal to

$$1 + \sum_{m=1}^{N} \varphi(m),$$

where $\varphi(m)$, *the Euler ϕ-function*, is the number of positive integers not exceeding m and relatively prime to m. Thus the density in question can be rewritten as

$$\rho = \lim_{N\to\infty} \frac{1 + \sum_{m=1}^{N} \varphi(m)}{(N+1)(N+2)/2} = 2 \lim_{N\to\infty} \frac{1 + \sum_{m=1}^{N} \varphi(m)}{N^2} \ . \tag{2}$$

The Euler function has the well known expression

$$\varphi(m) = m \prod_{p|m} \left(1 - \frac{1}{p}\right),$$

where the product is taken over the prime divisors of m. It is possible to use this expression for computing the limit in (2) directly, but we prefer another way to prove the Density Theorem, one which we believe requires a far less technical argument.

Our approach uses a slightly different expression for the density:

$$\rho = \lim_{N\to\infty} \frac{V(N)}{N^2} \ , \tag{3}$$

where $V(N)$ is the number of visibles in the square $1 \le x, y \le N$, that is,

$$V(N) = \left| \left\{ (m,n) \in \mathbb{Z}^2 : 1 \le m, n \le N,\ \gcd(m,n) = 1 \right\} \right| . \tag{4}$$

The equivalence of (3) to (1) and (2) is an easy exercise. To prove that the limit in (3) is equal to $6/\pi^2$, we need some preparation.

The Möbius function

In this section we discuss a classical tool from number theory. The *Möbius function* $\mu : \mathbb{Z}^+ \to \mathbb{Z}$ is defined recursively by $\mu(1) = 1$, and $\sum_{d|n} \mu(d) = 0$ for $n \ge 2$.

It is not difficult to obtain the following explicit formula for μ (exercise):

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1; \\ 0 & \text{if } n \text{ is divisible by a square;} \\ (-1)^k & \text{if } n \text{ is the product of } k \text{ distinct primes.} \end{cases} \tag{5}$$

The basic result about μ is the following.

Möbius Inversion Formula. *Let f and F be functions from $\mathbb{Z}^+ \to \mathbb{Z}$ that are zero for all but a finite number of integers, and are related by*

$$F(n) = \sum_{k \ge 1} f(nk)$$

for all n. Then f can be recovered from F by

$$f(n) = \sum_{d \ge 1} \mu(d) F(nd).$$

Proof. We have

$$\sum_{d \ge 1} \mu(d) F(nd) = \sum_{d \ge 1} \mu(d) \sum_{k \ge 1} f(ndk) = \sum_{m \ge 1} f(nm) \sum_{d|m} \mu(d)$$

(all these sums are finite). But, by definition of μ, we have $\sum_{d|m} \mu(d) = 0$ unless $m = 1$, so the last double sum reduces to $f(n)$. ■

(We note that the form of Möbius inversion most often found in number theory books says that if f and F are related by $F(n) = \sum_{d|n} f(d)$, then $f(n) = \sum_{d|n} \mu(d) F(n/d)$. The proof of this is very similar to the one above.)

Proof of the Density Theorem

Now we are ready to attack the limit in (3). We start by observing that, for every positive integer d, there is a one-to-one correspondence between the sets

$$\begin{aligned}&\{(m,n)\in\mathbb{Z}^2 : 1\le m,n\le N,\ \gcd(m,n)=d\}\ \text{ and}\\ &\{(m,n)\in\mathbb{Z}^2 : 1\le m,n\le \lfloor N/d\rfloor,\ \gcd(m,n)=1\}\end{aligned}$$

given by $(m,n)\to(m/d,n/d)$. (Here $\lfloor r\rfloor$ denotes the greatest integer not exceeding r.) Counting the lattice points (m,n) in the $N\times N$ square and grouping them according to the value of $\gcd(m,n)$, we conclude that

$$N^2=\sum_{d\ge1}V(\lfloor N/d\rfloor). \tag{6}$$

Now let us fix N (temporarily) and let

$$F(n)=\lfloor N/n\rfloor^2 \ \text{ and } f(n)=V(\lfloor N/n\rfloor).$$

Rewriting (6) with $\lfloor N/n\rfloor$ in place of N, and using the fact that $\left\lfloor\frac{\lfloor N/n\rfloor}{d}\right\rfloor=\left\lfloor\frac{N}{nd}\right\rfloor$, we see that these two functions are related by

$$F(n)=\sum_{d\ge1}f(nd)$$

(and this makes sense since $F(n)=0=f(n)$ for $n>N$). The Möbius Inversion Formula now tells us that

$$f(n)=\sum_{d\ge1}\mu(d)F(nd).$$

In particular, for $n=1$,

$$V(N)=f(1)=\sum_{d\ge1}\mu(d)\lfloor N/d\rfloor^2. \tag{7}$$

(Note that this sum need only be taken up to $d=N$ since $\lfloor N/d\rfloor=0$ for larger values of d.)

As a first step towards the proof of the Density Theorem, we use (7) to express the limit in (3) as the sum of an infinite series.

Proposition 4. $\displaystyle\lim_{N\to\infty}\frac{V(N)}{N^2}=\sum_{d\ge1}\frac{\mu(d)}{d^2}.$

Proof. We first note that the series $\sum_{d\ge1}\mu(d)/d^2$ converges absolutely since, from (5), $|\mu(d)|\le1$ for all d. Let

$$R_N=\left(\sum_{d=1}^{N}\frac{\mu(d)}{d^2}\right)-\frac{V(N)}{N^2}.$$

We proceed to show that $R_N\to0$ as $N\to\infty$. Using (7), we write

$$R_N=\sum_{d=1}^{N}\mu(d)\left(\frac{1}{d^2}-\frac{\lfloor N/d\rfloor^2}{N^2}\right).$$

Using $|\mu(d)| \le 1$ again:

$$|R_N| \le \sum_{d=1}^{N} \left(\frac{1}{d^2} - \frac{\lfloor N/d \rfloor^2}{N^2}\right) = \sum_{d=1}^{N} \left(\frac{1}{d} - \frac{\lfloor N/d \rfloor}{N}\right)\left(\frac{1}{d} + \frac{\lfloor N/d \rfloor}{N}\right).$$

Dividing the obvious inequality $0 \le \frac{N}{d} - \left\lfloor \frac{N}{d} \right\rfloor < 1$ by N and applying the result to the two factors on the right, we obtain the following upper bound for $|R_N|$:

$$|R_N| \le \sum_{d=1}^{N} \underbrace{\left(\frac{1}{d} - \frac{\lfloor N/d \rfloor}{N}\right)}_{<\frac{1}{N}} \underbrace{\left(\frac{1}{d} + \frac{\lfloor N/d \rfloor}{N}\right)}_{\le \left(\frac{1}{d}+\frac{1}{d}\right)} \le \frac{2H_N}{N},$$

where $H_N = \sum_{d=1}^{N} \frac{1}{d}$ is the Nth *harmonic number*. It is well-known that $\lim_{N\to\infty} H_N/N = 0$; the easiest way to see this is to use the rectangular approximation of the area under $y = 1/x$ as in Figure 5.

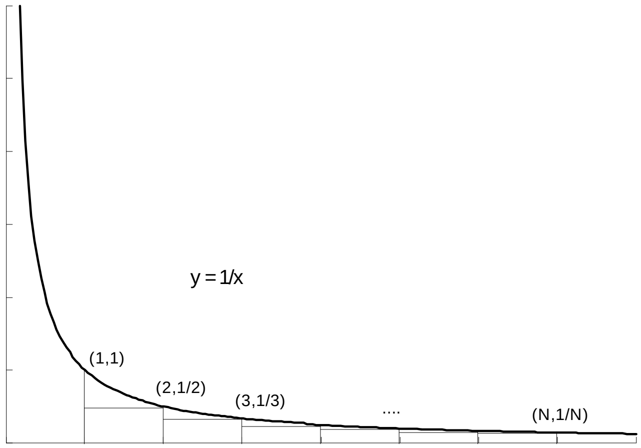

Figure 5. $\sum_{d=1}^{N} \frac{1}{d} \le 1 + \int_1^N \frac{dx}{x} = 1 + \ln N$.

This shows that $H_N \le 1 + \int_1^N \frac{dx}{x} = 1 + \ln N$. Since $\lim\limits_{N\to\infty} \dfrac{1+\ln N}{N} = 0$, we see that $\lim_{N\to\infty} R_N = 0$, as desired. ■

We are almost there: we now know that the density ρ is $\sum\limits_{d\ge 1} \frac{\mu(d)}{d^2}$; the next result tells us that it is just the reciprocal of a simpler series.

Proposition 5. $\left(\sum\limits_{d\ge 1} \frac{\mu(d)}{d^2}\right)\left(\sum\limits_{n\ge 1} \frac{1}{n^2}\right) = 1.$

Proof. Since each of the factors is an absolutely converging series, we can transform the product as follows:

$$\left(\sum_{d\geq 1}\frac{\mu(d)}{d^2}\right)\left(\sum_{n\geq 1}\frac{1}{n^2}\right)=\sum_{\substack{n\geq 1\\ d\geq 1}}\frac{\mu(d)}{d^2n^2}=\sum_{m\geq 1}\frac{1}{m^2}\sum_{d|m}\mu(d).$$

By definition of the Möbius function, $\sum_{d|m}\mu(d)=0$ unless $m=1$, so the final sum is simply 1.

■

The last result we need is one of the most famous calculations in mathematics, first made (non-rigorously) by Euler.

Proposition 6. $\sum_{n\geq 1}\frac{1}{n^2}=\frac{\pi^2}{6}$.

There are probably dozens of different proofs of this fact. The elementary ones use tricky integration by parts and Taylor series; others use complex variables and residue theory or double integrals. For a summary see [**6**]. We sketch a proof using yet another branch of mathematics, Fourier series (see [**7**] for an early similar account). We look at the odd function $f(x)=x$ and calculate the inner product $\langle f,f\rangle=\int_{-\pi}^{\pi}f(x)\cdot f(x)\ dx$ in two ways. On the one hand we have $\langle x,x\rangle=\int_{-\pi}^{\pi}x^2dx=\frac{2\pi^3}{3}$. On the other hand, we can expand f as a Fourier series in the basis of odd functions $\sin(nx)$ for $n\geq 1$. An easy and standard check shows that

$$\langle\sin(mx),\sin(nx)\rangle=\int_{-\pi}^{\pi}\sin(mx)\sin(nx)\ dx=\pi \text{ if } m=n \text{ and } 0 \text{ otherwise,}$$

and so the coefficient of $\sin(nx)$ in f is

$$\begin{aligned}a_n&=\frac{\langle x,\sin nx\rangle}{\langle\sin nx,\sin nx\rangle}=\frac{\langle x,\sin nx\rangle}{\pi}\\&=\frac{1}{\pi}\int_{-\pi}^{\pi}x\sin(nx)dx=\frac{1}{\pi}\int_{-\pi}^{\pi}xd\left(\frac{-\cos(nx)}{n}\right)\\&=-2\frac{\cos(n\pi)}{n}+\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\cos(nx)}{n}dx\\&=-2\frac{\cos(n\pi)}{n}=(-1)^{n-1}\frac{2}{n}.\end{aligned}$$

We conclude that

$$\frac{2\pi^3}{3}=\langle x,x\rangle=\left\langle\sum_{n\geq 1}a_n\sin(nx),\sum_{n\geq 1}a_n\sin(nx)\right\rangle=\pi\sum_{n\geq 1}a_n^2=4\pi\sum_{n\geq 1}\frac{1}{n^2},$$

and our result follows. ■

This concludes the proof of the Density Theorem. The above arguments are close to those in [**1**], where the Density Theorem is extended from the plane $\mathbb{R}^2$ to an arbitrary Euclidean space

$\mathbb{R}^n$; a similar proof also appears in [**9**, pp.105-108]. To state the more general theorem, we note that the series $\sum_{n\geq 1} \frac{1}{n^2}$ is a special case of the Riemann zeta function

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} \ .$$

The theorem for $\mathbb{R}^n$ states: if a lattice point is picked at random in n dimensions, the probability that it is visible from the origin is $1/\zeta(n)$.

The (complex) zeros of $\zeta(s)$ are the subject of the Riemann Hypothesis; now that Fermat's Last Theorem has finally been proved, this is considered by many to be the most important open problem in the whole of mathematics (see [**3**] for a fascinating account, as well as [**4**], the standard reference on the zeta function).

Densities and the Product Formula

As we have already mentioned, the definition of density (or "probability") of visibles has to be approached with care. For instance, despite the fact that the density in question ($6/\pi^2$) is rather high, there are arbitrarily large regions in $\mathbb{R}^2$ that do not contain *any* visibles. To be more precise, we have the following.

Proposition 7. *For every positive integer N, there exists a lattice point (a, b) for which the square $\{(x, y) : a - N \leq x < a,\ b - N \leq y < b\}$ contains no visibles.*

Proof. We use the well-known Chinese Remainder Theorem (see, for example, [**9**] or [**10**]): if $p_1, p_{2,} \ldots, p_M$ are distinct primes, and $r_1, r_{2,} \ldots, r_M$ are arbitrary integers, then there exists $a \in \mathbb{Z}$ such that $a \equiv r_i \bmod p_i$ for $i = 1, 2, \ldots, M$. For a given N, consider an $N \times N$ matrix (p_{ij}) whose entries are distinct prime numbers. By the Chinese Remainder Theorem, there exists $a \in \mathbb{Z}$ such that $a \equiv i \bmod p_{ij}$ for $i, j = 1, \ldots, N$; similarly, there exists $b \in \mathbb{Z}$ such that $b \equiv j \bmod p_{ij}$ for $i, j = 1, \ldots, N$. The corresponding lattice point (a, b) has the desired property; indeed, if $1 \leq i, j \leq N$, then both integers $a - i$ and $b - j$ are divisible by p_{ij}, and hence $(a - i, b - j)$ is *not* a visible. ■

For example, if we let $a = 22$ and $b = 176$ and use the matrix of primes $\begin{pmatrix} 7 & 3 \\ 5 & 2 \end{pmatrix}$, we get the square of non-visible points $(20, 174)$, $(21, 174)$, $(20, 175)$ and $(21, 175)$. Note that even this small 2×2 square requires fairly large values of a and b; in fact, as N increases, the magnitude of the numbers in the $N \times N$ block of non-visibles gets very large. (The authors would like to thank the referee for suggesting that we include this pretty application of a classical result from elementary number theory.)

We conclude by considering another "probabilistic" model for the density of visibles. Namely, for a "random" positive integer m and a prime p we assume that the p choices for the remainder

$m \mod p$ are equally likely. We also assume that, for different primes, the choices are independent of each other. Now let us consider $\gcd(m,n)$ for two random positive integers. There are p^2 possible remainders when each is divided by p, but only one of these — both remainders zero — gives $p \mid \gcd(m,n)$. So the probability that $\gcd(m,n)$ is *not* divisible by p is $(p^2-1)/p^2$. The independence assumption implies that the probability that m and n are relatively prime (that is, that (m,n) is a visible lattice point) can be expressed as the infinite product

$$\prod_{p \text{ prime}} \left(1 - \frac{1}{p^2}\right).$$

Remarkably, this product is also equal to $6/\pi^2$. To see why, observe that when computing it you get terms of the form $(-1)^k/(p_1^2 p_2^2 \cdots p_k^2)$ where the p_i are *distinct* primes. By the explicit formula (5) for the Möbius function, the probability that m and n are relatively prime is

$$\prod_{p \text{ prime}} \left(1 - \frac{1}{p^2}\right) = \sum (-1)^k/(p_1^2 p_2^2 \cdots p_k^2) = \sum_{d \geq 1} \frac{\mu(d)}{d^2} = \frac{6}{\pi^2},$$

the last equality being a consequence of Propositions 5 and 6.

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