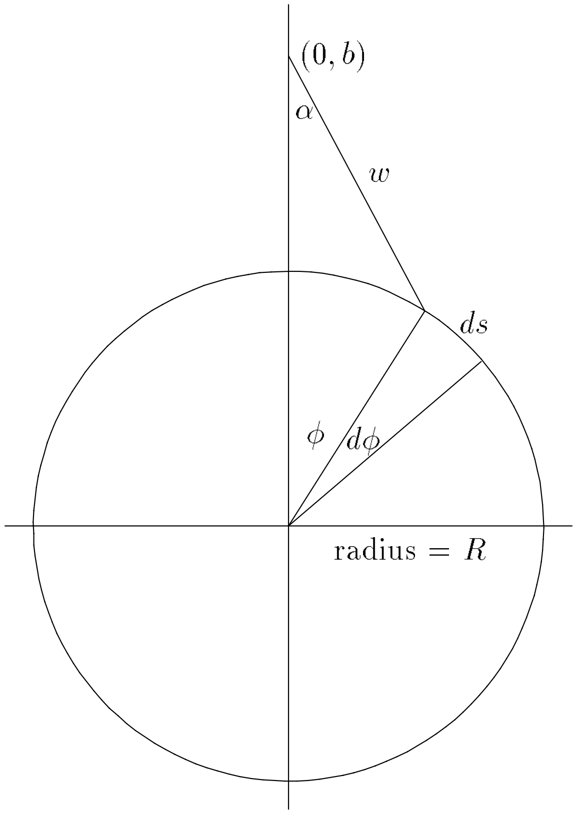
THE GRAVITATIONAL ATTRACTION OF A SPHERE



The small bit of arclength is $ds = R\,d\phi$. When rotated around the y-axis, this gives a narrow strip whose area is $dA = (2\pi R\sin(\phi))R\,d\phi = 2\pi R^2\sin(\phi)\,d\phi$. We suppose the whole sphere has mass M, so its area density is $\dfrac{M}{4\pi R^2}$. Thus, the small strip has mass $\dfrac{M}{4\pi R^2}\,dA = \dfrac{M}{2}\sin(\phi)\,d\phi$. Now the force on the mass m at $(0,b)$ due to this mass, by symmetry, has only a vertical component, which is $\cos(\alpha)$ times the total force due to the strip. By Newton's Law of Gravity, this is:

$$dF = \frac{mG}{w^2}\cdot\left(\frac{M}{2}\sin(\phi)\cos(\alpha)\right)\,d\phi.$$

Now we rewrite the last equation above so that it is a function of the distance w only. To do this, we use the Law of Cosines twice:

$$w^2 = R^2 + b^2 - 2Rb\cos(\phi)$$
$$R^2 = b^2 + w^2 - 2bw\cos(\alpha)$$

We differentiate the first to get $2w\,dw = 2Rb\sin(\phi)\,d\phi$, so $\sin(\phi)\,d\phi = \dfrac{w\,dw}{Rb}$.

We use the second equation to solve for $\cos(\alpha) = \dfrac{b^2+w^2-R^2}{2bw}$.

We now substitute these into the equation for dF and integrate to find the total force for $W_1 \le w \le W_2$:

$$F = \frac{MmG}{2}\int_{W_1}^{W_2}\frac{1}{w^2}\left(\frac{b^2+w^2-R^2}{2bw}\right)\frac{w\,dw}{Rb} = \frac{MmG}{4Rb^2}\int_{W_1}^{W_2}\left(\frac{b^2-R^2}{w^2}+1\right)\,dw$$
$$= \frac{MmG}{4Rb^2}\left[\left(\frac{R^2-b^2}{w}\right)+w\right]_{W_1}^{W_2}$$

When the point is outside the sphere (as in the diagram), $0 \leq R \leq b$, $W_1 = b - R$ and $W_2 = b + R$. This gives $F = \frac{MmG}{b^2}$; i.e. the same force as if the entire mass were located at the center of the sphere.

When the point is inside the sphere, $W_1 = R - b$, $W_2 = R + b$, and $F = 0$. The gravitational force inside a sphere is always 0.