Solution to practice final problems:

1. Find the derivative $f'(x)$ using the <u>definition</u> of the derivative. You must show all steps of your limit calculation.

(a) $f(x) = 5x^2 - 3x + 1$ **(b)** $f(x) = \dfrac{2x}{5x-2}$

<u>Solution</u>: **(a).**

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{5(x+h)^2 - 3(x+h) + 1 - [5x^2 - 3x + 1]}{h}$$

$$= \lim_{h\to 0} \frac{5(x^2 + 2xh + h^2) - 3x - 3h + 1 - 5x^2 + 3x - 1}{h} = \lim_{h\to 0} \frac{5x^2 + 10xh + 5h^2 - 3x - 3h + 1 - 5x^2 + 3x - 1}{h}$$

$$= \lim_{h\to 0} \frac{10xh + 5h^2 - 3h}{h} = \lim_{h\to 0} (10x + 5h - 3)$$

$$= 10x - 3$$

(b).

$$f'(x) = \lim_{h\to 0} \frac{\dfrac{2x+2h}{5x+5h-2} - \dfrac{2x}{5x-2}}{h} = \lim_{h\to 0} \frac{\dfrac{(2x+2h)(5x-2) - 2x(5x+5h-2)}{(5x+5h-2)(5x-2)}}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{10x^2 + 10xh - 4x - 4h - (10x^2 + 10xh - 4x)}{(5x+5h-2)(5x-2)}}{h} = \lim_{h\to 0} \frac{\dfrac{-4h}{(5x+5h-2)(5x-2)}}{h}$$

$$= \lim_{h\to 0} \frac{-4}{(5x+5h-2)(5x-2)} = \frac{-4}{(5x-2)^2}$$

2. Find the following derivatives.

(a) $\dfrac{d}{dx} e^{\tan(\sin x)} = e^{\tan(\sin x)} \sec^2(\sin x) \cos x$

(b)

$$\frac{d}{dx} x \arctan(\sin\sqrt{x}) = (1)\arctan(\sin\sqrt{x}) + x\left[\frac{1}{1 + (\sin\sqrt{x})^2}(\cos\sqrt{x})\frac{1}{2\sqrt{x}}\right] = \arctan(\sin\sqrt{x}) + \frac{\sqrt{x}(\cos\sqrt{x})}{2[1 + (\sin\sqrt{x})^2]}$$

(c) $\dfrac{d}{dx}(x\cos x - e^{x^3 - 3e}) = (1)\cos x + x(-\sin x) - e^{x^3 - 3e}(3x^2 - 0) = \cos x - x\sin x - 3x^2 e^{x^3 - 3e}$

(d)

$$\frac{d}{dx}\left(\frac{\sqrt{x}}{5 + \cos x}\right) = \frac{\dfrac{1}{2\sqrt{x}}(5 + \cos x) - \sqrt{x}(0 - \sin x)}{(5 + \cos x)^2} = \frac{(5 + \cos x) - (2\sqrt{x})\sqrt{x}(-\sin x)}{2\sqrt{x}(5 + \cos x)^2} = \frac{5 + \cos x + 2x\sin x}{2\sqrt{x}(5 + \cos x)^2}$$

(e) $$\frac{d}{dx}\left(\frac{2^{\sin x}}{\ln x}\right) = \frac{(\ln 2)2^{\sin x}(\ln x) - 2^{\sin x}(\frac{1}{x})}{(\ln x)^2} = \frac{x(\ln 2)2^{\sin x}(\ln x) - 2^{\sin x}}{x(\ln x)^2} = \frac{2^{\sin x}[(\ln 2)x\ln x - 1]}{x(\ln x)^2}$$

(f) $\frac{d}{dx}(\ln e^{ax}+\ln b)=\frac{1}{e^{ax}}e^{ax}(a)+0=a$. Notice *a*, *b* are constants, so that (lnb)'=0

3. Find $\frac{dy}{dx}$ by logarithmic differentiation for:

(a) $y=\frac{x(x-10)^8e^x}{\sqrt{x-6}(x+14)^2}$

$$\ln y=\ln x+8\ln(x-10)+\ln(e^x)-\ln(\sqrt{x-6})-\ln[(x+14)^2]=\ln x+8\ln(x-10)+x-\frac{1}{2}\ln(x-6)-2\ln(x+14)$$

$$\frac{1}{y}y'=\frac{1}{x}+\frac{8}{x-10}+1-\frac{1}{2}(\frac{1}{x-6})-2(\frac{1}{x+14})$$

$$\frac{dy}{dx}=y'=\frac{x(x-10)^8e^x}{\sqrt{x-6}(x+14)^2}[\frac{1}{x}+\frac{8}{x-10}+1-\frac{1}{2(x-6)}-\frac{2}{x+14}]$$

(b) $y=x^{x+\ln x}$

$$\ln y=\ln(x^{x+\ln x})=(x+\ln x)\ln x$$

$$\frac{1}{y}y'=(1+\frac{1}{x})\ln x+(x+\ln x)\frac{1}{x}=\ln x+\frac{\ln x}{x}+\frac{x}{x}+\frac{\ln x}{x}=\ln x+\frac{2\ln x}{x}+1$$

$$\frac{dy}{dx}=y'=x^{x+\ln x}(\ln x+\frac{2\ln x}{x}+1)$$

4. Let $g(x)=e^{1+xf(x)}$. Find $g'(1)$ if $f(1)=-1$ and $f'(1)=2$. Find the equation of the tangent line to $y=g(x)$ at the point (1,1).

$$g'(x)=e^{1+xf(x)}[1+xf(x)]'=e^{1+xf(x)}[0+(1)f(x)+xf'(x)]=e^{1+xf(x)}[f(x)+xf'(x)]$$

$$g'(1)=e^{1+1f(1)}[f(1)+1f'(1)]=e^{1+(-1)}[-1+2]=e^0(1)=1$$

So tangent line is $y-1=1(x-1)$, that is $y=x$.

5. Find $\frac{dy}{dx}$ for $x^4y^2+2y=3x$. Find the equation of the tangent line to this curve at the point (1,1).

$$(4x^3)y^2+(x^4)2yy'+2y'=3$$

So $(2x^4y+2)y'=3-4x^3y^2$

$$\frac{dy}{dx}=y'=\frac{3-4x^3y^2}{2x^4y+2}.$$

The tangent slope is $\frac{dy}{dx}|_{x=1,y=1}=\frac{3-4}{2+2}=-\frac{1}{4}$.

The tangent line equation is $y-1=-\frac{1}{4}(x-1)$, that is $y=-\frac{1}{4}x+\frac{5}{4}$

6. For the parametric curve $x = \arcsin t, \quad y = t + 2$. Find the slope of the tangent line when t=0. Find the equation of the tangent line when t=0. Does the curve have horizontal tangent? Eliminate the parameter to find a Cartesian equation of the curve. Sketch the curve.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1}{1/\sqrt{1-t^2}} = \sqrt{1-t^2}\ .$$

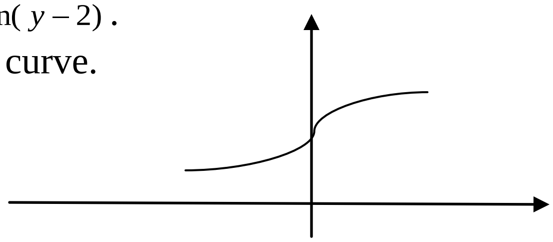
So the slope of tangent line when t=0 is $\frac{dy}{dx}|_{t=0} = \sqrt{1-0} = 1$.

When t=0, $x = \arcsin 0 = 0, \quad y = 0 + 2 = 2$, the curve pass point (0,2). So the tangent line equation is $y - 2 = 1(x - 0)$, that is $y = x + 2$.

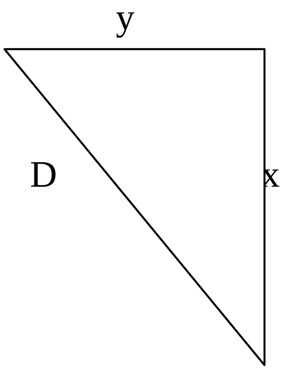
Horizontal tangent if the slope is zero, that is $\frac{dy}{dx} = \sqrt{1-t^2} = 0$, which occurs when t=1 or t=-1. That is, at the points (arcsin1,1+2)=(π/2, 3) and (arcsin(-1), -1+2)=(-π/2, 1).

Solve $t = y - 2$, so the Cartesian equation is $x = \arcsin(y-2)$.

Use your graphing calculator, you could sketch the curve. Notice there is an implicit domain restriction t in [-1,1] which lead to x in [-π/2, π/2] and y in [1,3]. So you should only get a segment of the curve for the sketch.



7. Cyclist A and B starts out at 1:00 PM from a tower and going away at two different directions. Cyclist A rides south at the constant speed of 20 miles per hour. Cyclist B rides west at the constant speed of 15 miles per hour. At 3:00 PM, what is the rate at which the distance between the cyclists is increasing?



Know $\frac{dx}{dt} = 20, \ \frac{dy}{dt} = 15$

want to find $\frac{dD}{dt}|_{t=2}$

The relationship is $x^2 + y^2 = D^2$

So $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2D\frac{dD}{dt}$,

$$\frac{dD}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{D} = \frac{20x + 15y}{D}$$

When t=2, x=20(2)=40, y=15(2)=30 and $D = \sqrt{x^2 + y^2} = \sqrt{40^2 + 30^2} = 50$.

$\frac{dD}{dt}|_{t=2} = \frac{20(40) + 15(30)}{50} = 25$ miles per hour.

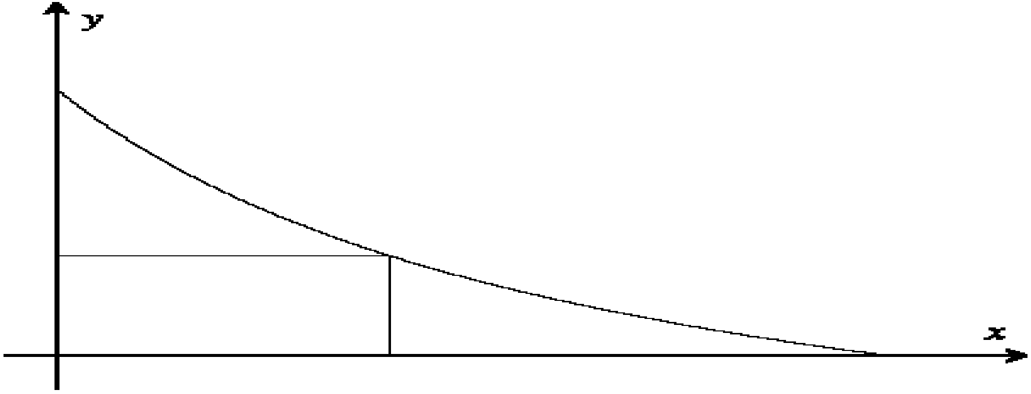
8. Set up Newton's method formula applied to the equation $x^3 - 5 = 0$. Find the third approximations x_3 for the root if the first approximation is $x_1 = 2$.

$f(x) = x^3 - 5$, so $f'(x) = 3x^2$.

The Newton's method formula is $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} = x_n - \dfrac{(x_n)^3 - 5}{3(x_n)^2}$.

$x_2 = 2 - \dfrac{(2)^3 - 5}{3(2)^2} = 2 - \dfrac{8-5}{12} = 1.75$, $x_3 = 1.75 - \dfrac{(1.75)^3 - 5}{3(1.75)^2} = 1.710884354$.

9. A rectangle is contained between the x-axis, the y-axis and the curve $y = \dfrac{3-x}{x+2}$ as shown in the graph below. Find the largest possible area of the rectangle.



Wants to maximize the area $A = xy$.

x and y are related by the curve $y = \dfrac{3-x}{x+2}$.

So we are maximizing $A(x) = x(\dfrac{3-x}{x+2}) = \dfrac{3x - x^2}{x+2}$ on the interval [0, 3].

To find the critical number, set $A'(x) = 0$.

Since $A'(x) = \dfrac{(3-2x)(x+2) - (3x - x^2)(1)}{(x+2)^2} = \dfrac{3x - 2x^2 + 6 - 4x - 3x + x^2}{(x+2)^2} = \dfrac{-x^2 - 4x + 6}{(x+2)^2}$,

$-x^2 - 4x + 6 = 0$, $-(x^2 + 4x - 6) = 0$, $x^2 + 4x - 6 = 0$

$x = \dfrac{-4 \pm \sqrt{4^2 - 4(1)(-6)}}{2(1)} = \dfrac{-4 \pm \sqrt{40}}{2} = -2 \pm \sqrt{10}$. We only need to consider the positive critical number $x = -2 + \sqrt{10}$ here.

For critical number $x = -2 + \sqrt{10}$, $A(-2 + \sqrt{10}) = \dfrac{3x - x^2}{x+2} = 0.6754$

For endpoints $x = 0$, $A(0) = \dfrac{0}{2} = 0$; $x = 3$, $A(3) = \dfrac{9-9}{5} = 0$.

So the maximum area is $A(-2 + \sqrt{10}) = \dfrac{3x - x^2}{x+2} = 0.6754$.

10. *derivative* $f'(x) = \frac{x}{x^2+1}$

(a) Find the intervals of increasing for the function $f(x)$.

(b) Find the interval on which the function $f(x)$ concave upwards.

(c) Find the x values of local maxima, local minima and inflection points.

(a) $f'(x) = \frac{x}{x^2+1}$ is positive for $x > 0$ and negative for $x < 0$.

Hence the interval of increase is $(0, \infty)$ and the interval of decrease is $(-\infty, 0)$.

(b) $f''(x) = \frac{1(x^2+1) - x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$. So $f''(x) = 0$ for x=1 or x=-1. These two points separate the real line into three intervals $(-\infty,-1)$, $(-1,1)$ and $(1,\infty)$.

On $(-\infty,-1)$, $f''(x) = \frac{1-x^2}{(x^2+1)^2} < 0$; on $(-1,1)$, $f''(x) = \frac{1-x^2}{(x^2+1)^2} > 0$; on $(1,\infty)$, $f''(x) > 0$.

So the function concaves upwards on intervals $(-\infty,-1)$ and $(1,\infty)$.

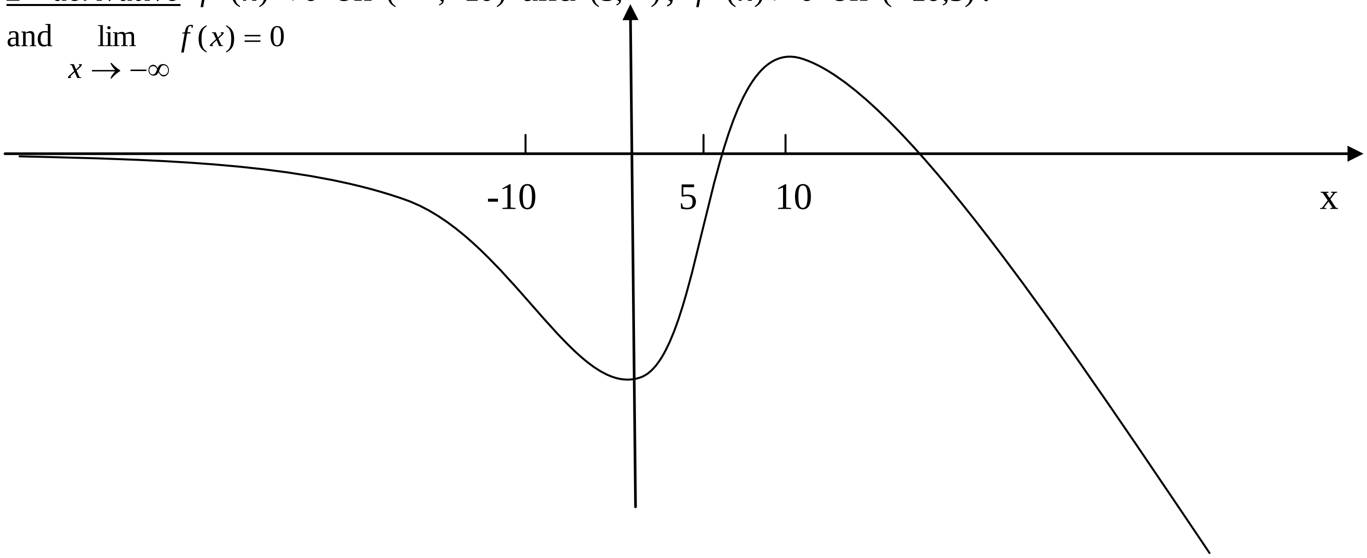
(c) From (a) we know that x=0 is the local minimum (decrease before x=0 and increase afterwards); there is no local maximum. From (b) we know that x=1 and x=-1 are inflection points.

11. Sketch the graph of a function with

derivative $f'(x) < 0$ on $(-\infty, 0)$ and $(10, \infty)$; $f'(x) > 0$ on $(0,10)$;

2nd derivative $f''(x) < 0$ on $(-\infty,-10)$ and $(5,\infty)$; $f''(x) > 0$ on $(-10,5)$.

and $\lim_{x \to -\infty} f(x) = 0$



12. Calculate the Riemann sum for $\int_0^{\pi} \sin x dx$ with n=2, using **(a)** right endpoints; **(b)** left endpoints; **(c)** midpoints. Which of the three answers above is closest to the extra value of the integral?

$$\Delta x = \frac{\pi - 0}{2} = \frac{\pi}{2}$$

(a) $R_2 = \frac{\pi}{2}[f(\frac{\pi}{2}) + f(\pi)] = \frac{\pi}{2}[\sin\frac{\pi}{2} + \sin\pi] = \frac{\pi}{2}[1+0] = \frac{\pi}{2} = 1.57$;

(b) $R_2 = \frac{\pi}{2}[f(0) + f(\frac{\pi}{2})] = \frac{\pi}{2}[\sin 0 + \sin\frac{\pi}{2}] = \frac{\pi}{2} = 1.57$;

(c) $R_2 = \frac{\pi}{2}[f(\frac{\pi}{4}) + f(\frac{3\pi}{4})] = \frac{\pi}{2}[\sin\frac{\pi}{4} + \sin\frac{3\pi}{4}] = \frac{\pi}{2}[\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}] = \frac{\sqrt{2}\pi}{2} = 2.22$.

$\int_0^{\pi} \sin x dx = -\cos x |_0^{\pi} = -\cos\pi - (-\cos 0) = 2$. The Riemann sum using midpoints in (c) is closest.

13. Find $f(x)$ for **(a)** $f'(x) = 3^x + 2\sin x$, $f(x) = \frac{3^x}{\ln 3} - 2\cos x + C$;

(b) $f'(x) = \frac{1}{\cos^2 x} - \sqrt{x}$,

rewrite $f'(x) = \sec^2 x - x^{\frac{1}{2}}$, so $f(x) = \tan x + \frac{2}{3}x^{\frac{3}{2}} + C$;

(c) $f'(x) = \frac{\ln x}{x} - 7x^2 + 2$,

let $u = \ln x$, $\frac{du}{dx} = \frac{1}{x}$, $dx = xdu$, $\int \frac{\ln x}{x} dx = \int \frac{u}{x} xdu = \int udu = \frac{u^2}{2} + C$,

so $f(x) = \frac{(\ln x)^2}{2} - \frac{7}{3}x^3 + 2x + C$.

14. (a) $\int_0^{\pi/3} \cos(2x)dx$, $u = 2x$, $\frac{du}{dx} = 2$, $dx = \frac{1}{2}du$,

$$\int_{x=0}^{\pi/3} \cos(u)\frac{1}{2}du = \frac{1}{2}\sin u |_{x=0}^{\pi/3} = \frac{1}{2}\sin(2x)|_{x=0}^{\pi/3} = \frac{1}{2}\sin(\frac{2\pi}{3}) - \frac{1}{2}\sin(0) = 0.433$$

(b) $\int (x^e + e^x)dx = \frac{x^e}{e+1} + e^x + C$