Scores by problem:

1	2	3	4	5	6	7	8	9	10	Total

Instructor: ______________________ **Name:** ______________________________

NORTHEASTERN UNIVERSITY
MTH U241 (Calculus I) **Final Exam** **Fall 2008**

SHOW YOUR WORK. There is NO credit for the answer without the work shown. If there is not enough room to show your work, use the back of the preceding page. Where a blank is provided for your answer, you must place your final answer in that blank to receive credit.

1. (a) For the function $f(x) = \dfrac{1}{2x+1}$ use the definition of the derivative to write $f'(0)$ as the limit of an algebraic expression, and compute the limit by simplifying this algebraic expression. **(5 pts)**

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{2h+1} - 1}{h}$$

$$= \lim_{h \to 0} \frac{\frac{\cancel{1} - 2h - \cancel{1}}{2h+1}}{h}$$

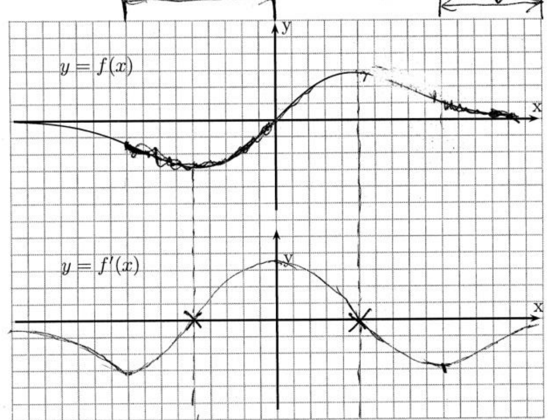
$$= \lim_{h \to 0} \frac{-2}{2h+1}$$

$$= \frac{-2}{0+1}$$

$$= -2$$

intervals of concaving up

(b) The graph of function f is given below. Use it to sketch the graph of its derivative f'. On the graph of f indicate (by thickening the corresponding part of the graph) the points where the graph of f is concave upwards. **(5 pts)**



The important thing is to realize local max/min on your $f'(x)$ graph corresponds to inflection points on $f(x)$

local min $f' = 0$ local max $f' = 0$

2. Calculate the following derivatives. If possible, simplify your answer.

(a) **(2 pts)** $\frac{d}{du}\left[\frac{u \tan u + u^2}{u}\right] =$ $\sec^2 u + 1$

$= [\tan u + u]'$

(b) **(2 pts)** $\frac{d}{dx}\left[\frac{\sin x}{\sin x+1}\right] =$ $\underline{\frac{\cos x}{(\sin x+1)^2}}$

$$\frac{\cos x(\sin x+1)-(\sin x)\cos x}{(\sin x+1)^2}$$

(c) **(3 pts)** $\frac{d}{dt}\left[\arctan(2t)-\ln(4t^2+1)\right] =$ $\underline{\frac{2-8t}{1+4t^2}}$

$$\frac{1}{1+(2t)^2}(2)-\frac{1}{4t^2+1}(8t)$$

$$=\frac{2}{1+4t^2}-\frac{8t}{4t^2+1}$$

(d) **(3 pts)** $\frac{d}{dx}\left[x^{1/x}\right] =$ $\underline{x^{\frac{1}{x}}\frac{(1-\ln x)}{x^2}}$

$$y = x^{\frac{1}{x}}$$

$$\ln y = \frac{1}{x}\ln x$$

$$\frac{1}{y}y' = \left(-\frac{1}{x^2}\right)\ln x+\frac{1}{x}\left(\frac{1}{x}\right)$$

$$\therefore y' = y\left[-\frac{\ln x}{x^2}+\frac{1}{x^2}\right] = y\left[\frac{1-\ln x}{x^2}\right]$$

3. Consider the curve defined by: $x^3 + y^3 = 2\ln y$.

(a) Use implicit differentiation to find the expression for dy/dx as a function of x and y. **(3 pts)**

Answer: $dy/dx =$ $-3x^2/(3y^2-2/y)$

$$3x^2 + 3y^2 \cdot y' = 2\,\frac{1}{y}\cdot y'$$

$$\left(3y^2 - \frac{2}{y}\right)y' = -3x^2$$

(b) Find the equation of the tangent line to the curve at $(-1, 1)$. **(4 pts)**

The tangent line is: $y = -3x - 2$

$$m = \frac{-3(-1)^2}{3(1)^2 - \frac{2}{1}} = \frac{-3}{3-2} = -3$$

$$\therefore\ y - 1 = -3(x - (-1))$$

$$y = -3x - 3 + 1$$

$$y = -3x - 2$$

(c) Use the linear approximation of y at $x = -1$ to estimate the y-coordinate of the point on the curve with x-coordinate being -1.1. **(3 pts)**

Answer: $y(-1.1) \approx$ 1.3

$$y \approx -3(-1.1) - 2 = 3.3 - 2 = 1.3$$

4. For the function $y = f(x) = x^2 e^x$

(a) Find all critical numbers for $f(x)$. For each critical number determine whether it is a local maximum, local minimum, or none of the above. (**3 pts**)

The critical points, with their types, are

$x = 0$ local min, $\quad x = -2$ local max

$$f'(x) = (2x)e^x + x^2(e^x) = x(2+x)e^x$$

$$\therefore \quad f' = 0 \iff x(2+x)e^x = 0$$

$$x = 0 \text{ and } x = -2.$$

$$f''(x) = [(2)e^x + (2x)(e^x)] + [(x^2)e^x + (2x)e^x]$$

$$= (x^2 + 4x + 2)e^x$$

$$f''(0) = 2e^0 > 0 \quad \therefore \quad x = 0 \text{ is local min}$$

$$f''(-2) = (4 - 8 + 2)e^{-2} = -2e^{-2} < 0 \quad \therefore \quad x = -2 \text{ is local max}$$

(b) Find intervals of increase/decrease for $f(x)$. (**2 pts**)
Make sure your answer is completely clear, complete, and unambiguous.

$f'(x)$ $\quad +$ $\quad -$ $\quad +$

$\quad\quad -2 \quad\quad 0$

$f(x)$ $\quad \nearrow$ $\quad \searrow$ $\quad \nearrow$

$\therefore$ $f(x)$ increases on $(-\infty, -2)$ and $(0, \infty)$

$f(x)$ decreases on $(-2, 0)$

(c) Find the x-coordinates of all the inflection points for $f(x)$. **(2 pts)**

The inflection points are : $-2+\sqrt{2}$, $-2-\sqrt{2}$

$$(x^2+4x+2)e^x = 0$$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-4 \pm \sqrt{16-8}}{2(1)}$$

$$= \frac{-4 \pm \sqrt{8}}{2}$$

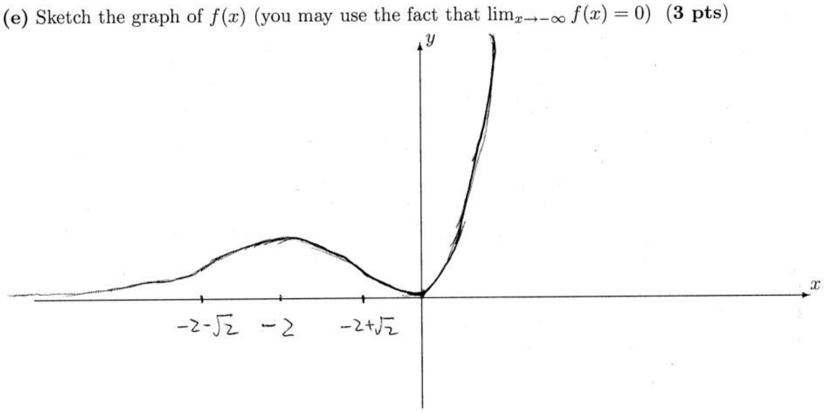
$$= -2 \pm \sqrt{2}$$

$$f''(x) = [x-(-2+\sqrt{2})][x-(-2-\sqrt{2})]\, e^x$$

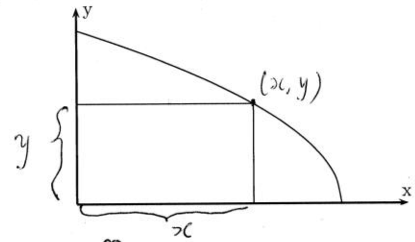
clearly change signs at the two roots.

$\therefore$ Are inflection points

(e) Sketch the graph of $f(x)$ (you may use the fact that $\lim_{x \to -\infty} f(x) = 0$) **(3 pts)**



5. A rectangle is contained between the x-axis, y-axis and the curve $y = \sqrt{12 - 2x}$ as shown in the graph. What is the largest possible area of the rectangle? **(10 pts)**



The largest possible area is: 8

Maximize Area

$$A = x \cdot y$$

Curve $y = \sqrt{12-2x}$

$\therefore$ Maximize $A(x) = x\sqrt{12-2x}$

over $0 \le x \le 6$ $\rightarrow$ $y = \sqrt{12-2x} = 0$

$$A'(x) = (1)\sqrt{12-2x} + x\frac{1}{2\sqrt{12-2x}}(-2)$$

$$= \sqrt{12-2x} - \frac{x}{\sqrt{12-2x}}$$

$$A'(x) = 0 \iff \frac{12-2x}{\sqrt{12-2x}} - \frac{x}{\sqrt{12-2x}} = 0$$

$$\frac{12-3x}{\sqrt{12-2x}} = 0 \quad \therefore \quad 12 = 3x, \; x = 4$$

$x = 4$, $A(4) = 4\sqrt{4}$

$x = 0$, $A(0) = 0$

$x = 6$, $A(6) = 6\sqrt{0} = 0$

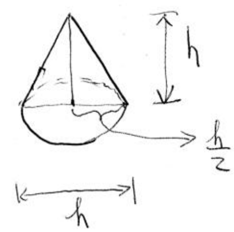
$\therefore$ Maximum at $x = 4$

$A(4) = 4\sqrt{4} = 4(2) = 8$

7

6. Gravel is being dumped from a conveyor belt at a rate of $\rho = 30$ ft^3/min, and its coarseness is such that it forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is 10 ft high? (**10 pts**)

(Recall that the volume of a cone of height h and base radius r is $\frac{\pi}{3}r^2h$.)



Know: $\frac{dV}{dt} = 30 \ \frac{ft^3}{min}$

Want: $\left.\frac{dh}{dt}\right|_{h=10} = ?$

Relation: $V = \frac{\pi}{3} r^2 h = \frac{\pi}{3}\left(\frac{h}{2}\right)^2 h$

$\therefore \ V = \frac{\pi}{12} h^3$

$\frac{dV}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{4}{\pi h^2} \frac{dV}{dt}$

$\left.\frac{dh}{dt}\right|_{h=10} = \frac{4}{100\pi}(30) = \frac{1.2}{\pi} \left(\frac{ft^3}{min}\right)$

7. Consider the parametric curve $x = 1/(t+1)^{1/2}$, $y = t/(t+1)$, $t > -1$

(a) Find dy/dx in terms of t. **(3 pts)**

Answer: $dy/dx =$ ____________________

$$\frac{dy/dt}{dx/dt} = \frac{\left(\frac{t}{t+1}\right)'}{\left(\frac{1}{\sqrt{t+1}}\right)'} = \frac{\frac{(1)(t+1)-t(1)}{(t+1)^2}}{-\frac{1}{2}(t+1)^{-\frac{3}{2}}(1)} = \frac{\frac{1}{(t+1)^2}}{-\frac{1}{2(t+1)^{\frac{3}{2}}}}$$

$$= -\frac{2}{\sqrt{t+1}}$$

(b) Use the value of dy/dx found in (a) to explain whether the curve has a vertical tangent. **(3 pts)**

$\frac{dy}{dx} = \infty$ if $\sqrt{t+1} = 0$

or $t = -1$

Since domain is $t > -1$,

this never happens.

(c) Eliminate the parameter t to find a Cartesian equation of the curve. **(4 pts)**

The equation is: $1/\sqrt{\frac{y}{1-y} + 1}$

$y = \frac{t}{t+1}$ $\therefore$ $ty + y = t$

$(y-1)t = -y$

$t = -\frac{y}{y-1} = \frac{y}{1-y}$

8. (a) Use Newton's method to determine the formula for the $(n+1)$ approximation x_{n+1} of the real root of $x^3 + x = 1$. Compute the second and third approximations, x_2 and x_3, if the first approximation is $x_1 = 0$. **(5 pts)**

$x_2 =$ 1 $\qquad\qquad\qquad\qquad$ $x_3 = \frac{3}{4}$

$x^3 + x - 1 = 0$

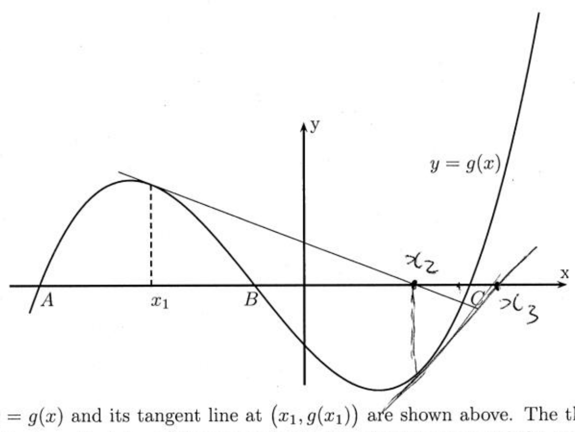
$f(x) = x^3 + x - 1$

$f'(x) = 3x^2 + 1$

$x_{n+1} = x_n - \frac{(x_n)^3 + x_n - 1}{3(x_n)^2 + 1}$

$x_2 = 0 - \frac{-1}{1} = 1$, $\quad x_3 = 1 - \frac{1+1-1}{3+1} = 1 - \frac{1}{4}$

(b) **(5 pts)**



The graph of $y = g(x)$ and its tangent line at $(x_1, g(x_1))$ are shown above. The three zeros of g are A, B, and C. Newton's method is used to approximate one of these zeros and the initial guess is x_1, as shown in the figure. The next two approximations are x_2 and x_3.

(i) Add to the figure above to show how x_2 and x_3 are determined. Label x_2 and x_3 on the figure above.

(ii) To which of the zeros, A, B, or C, will the initial approximation of x_1 lead? C

9. Let $f(x) = \dfrac{12}{(4+x)^2}$.

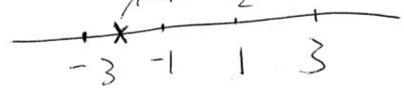
(a) Use the **midpoint rule** with a subdivision of the interval $[-3, 3]$ into three equal parts to **approximate**

$$\int_{-3}^{3} f(x)\,dx \approx \underline{\frac{49}{6}}.$$

Show your work in a clear, well-organized, detailed way. Failure to do so will cost points. Give your answer as a fraction in the blank provided above. Do not round.

$$\Delta x = \frac{3-(-3)}{3} = 2$$

$$\bar{x}_1 = \frac{-3+(-1)}{2} = -2$$



$$R_3 = 2\left[f(-2) + f(0) + f(2)\right]$$

$$= 2\left[\frac{12}{(4-2)^2} + \frac{12}{(4+0)^2} + \frac{12}{(4+2)^2}\right]$$

$$= 2\left[3 + \frac{3}{4} + \frac{1}{3}\right] = 2\left(3 + \frac{13}{12}\right) = \frac{49}{6}$$

(b) Use the Fundamental Theorem of Calculus to give the **exact** value for the same integral. Show your work.

$$\int_{-3}^{3} f(x)\,dx = \underline{\frac{72}{7}}.$$

$$u = 4x, \quad \frac{du}{dx} = 1$$

$$\therefore dx = du$$

$$\int_{x=-3}^{3} \frac{12}{u^2}\,du = 12\,\frac{u^{-1}}{-1}\Bigg|_{x=-3}^{3}$$

$$= -\frac{12}{4+x}\Bigg|_{-3}^{3}$$

$$= -\left(\frac{12}{7} - \frac{12}{1}\right) = -12\left(-\frac{6}{7}\right)$$

$$= \frac{72}{7}$$

10. Compute and simplify if possible

(a) **(3 pts)** $\int_1^4 \frac{u^3+u^2+1}{u^3}\,du =$ $3\frac{15}{32}+2\ln 2$

$$\int_1^4 \left(1+\frac{1}{u}+\frac{1}{u^3}\right)du$$

$$= \left[u+\ln u-\frac{1}{2u^2}\right]_1^4$$

$$= \left[4+\ln 4-\frac{1}{32}\right]-\left[1+\ln 1-\frac{1}{2}\right]$$

$$= 4-\frac{1}{32}-1+\frac{1}{2}+\ln 4-\ln 1$$

$$= 3\frac{15}{32}+2\ln 2-0$$

(b) **(2 pts)** $\int \frac{1+\ln t}{t}\,dt =$ $\frac{1}{2}(1+\ln t)^2+C$

$$u = 1+\ln t$$

$$\frac{du}{dt}=\frac{1}{t} \quad \therefore\ dt = t\,du$$

$$\int \frac{u}{\cancel{t}}\,t\,du = \frac{u^2}{2}+c$$

(c) **(2 pts)** $\int e^y \sin(e^y - 1)\, dy =$ $-\cos(e^y - 1) + C$

$u = e^y - 1$

$\frac{du}{dy} = e^y \quad \therefore dy = \frac{1}{e^y} du$

$\therefore \int \cancel{e^y} \sin(u) \frac{1}{\cancel{e^y}} du$

$= -\cos(u) + C$

(d) **(3 pts)** $\int_0^{1/2} \frac{1}{1+4t^2}\, dt =$ $\frac{\pi}{8}$

$\frac{1}{1+4t^2} = \frac{1}{1+(2t)^2}$

$u = 2t, \quad \frac{du}{dt} = 2$

$\therefore dt = \frac{1}{2} du$

$\int_0^{\frac{1}{2}} \frac{1}{1+u^2} \frac{1}{2} du = \frac{1}{2} \arctan u \Big|_{t=0}^{\frac{1}{2}}$

$= \frac{1}{2} \arctan(2t) \Big|_0^{\frac{1}{2}}$

$= \frac{1}{2} [\arctan 1 - \arctan 0]$

$= \frac{1}{2} (\frac{\pi}{4})$