Infinite Measure Preserving Odometers and Multiple Recurrence

by

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A family of infinite measure preserving odometers is presented which exhibit examples of p-recurrent but not $p+1$-recurrent ergodic transformations for every $p > 1$.

H. Furstenberg [6] proved that a finite measure preserving transformation is multiply recurrent. For infinite measure spaces the situation is different. It is possible for an ergodic measure preserving transformation on an infinite measure space to be multiply recurrent or not. Furthermore, for every $p > 1$ we will demonstrate the existence of ergodic, infinite measure preserving transformations which are p-recurrent but not $p+1$-recurrent (Theorem 3).

The examples will be members of a class of infinite measure preserving odometers. This type of construction has been studied by many authors ([1], [2], [8], [9], [10], [11], [12], [13]). In general, in this situation, a nonsingular product measure is given for the odometer and it is then proved that there exists an equivalent infinite invariant measure. This is done by analyzing either the ratio set (for example see [2]) or by showing the Radon-Nikodym derivative is 1 on an induced set (for example see [1] and [12]).

The construction in this paper is slightly simpler to analyze because of the existence of readily accessible exhaustive weakly wandering sets. The full space X will be a disjoint union of a countable number of subsets which arise naturally as the images of the exhaustive weakly wandering set. On each of these subsets the invariant measure will be equivalent to a product measure. Hence, in these examples, the infinite invariant measure is constructed explicitly.

The Odometers

In this section we present the basic notation and construction of the odometers. Each odometer T will depend upon a sequence $\mathcal{M}$ of positive integers from which the multiply recurrent properties can be derived (see Theorem 3. below). We exhibit a close connection between the multiply recurrent feature of T and the boundedness of a subsequence of $\mathcal{M}$.

This clarifies, with respect to multiply recurrence, the unusual behavior of these odometers. For completeness we present elementary proofs that the constructed odometers are ergodic and infinite measure preserving.

Let $\mathcal{M} = \{m_i \geq 2 \,|\, i \geq 0\}$ be a set of integers, and for each $i = 0, 1, 2, \ldots$ let $(\eta_{i,0}, \eta_{i,1}, \ldots, \eta_{i,m_i-1})$ be a set of m_i positive numbers with $\sum_{0 \leq j < m_i} \eta_{i,j} = 1$. For each $i \geq 0$ we consider the finite measure space $(\mathbf{Z}_i, \mathcal{B}_i, \mu_i)$, where $\mathbf{Z}_i$ consists of the m_i points $\{0, 1 \ldots, m_i - 1\}$, $\mathcal{B}_i$ is the set of all subsets of $\mathbf{Z}_i$, and the measure μ_i is defined by $\mu_i(j) = \eta_{i,j}$ for $0 \leq j < m_i$. It follows that the infinite direct product measure space $(\mathbf{Z}, \mathcal{B}, \mu) = \prod_{0 \leq i < \infty} (\mathbf{Z}_i, \mathcal{B}_i, \mu_i)$ is a finite measure space with $\mu(\mathbf{Z}) = 1$. In order to avoid unnecessary complications later, we eliminate from the set Z the countable sets

$$\bigcup_{0 \leq k < \infty} \{(x_0, x_1, , \ldots) \in Z \,|\, x_i = 0 \ for\ all\ i \geq k\}$$

and

$$\bigcup_{0 \leq k < \infty} \{(x_0, x_1, , \ldots) \in Z \,|\, x_i = 1 \ for\ all\ i \geq k\}.$$

For a point $x = (x_0, x_1, \ldots) \in Z$ and a positive integer $i \geq 0$ we shall say that x_i is the i-th coordinate of x. We also define the operation $+$ in $\mathbf{Z}$ as follows: for two members $x = (x_0, x_1, \ldots)$, $y = (y_0, y_1, \ldots) \in \mathbf{Z}$ we let $x + y = (z_0, z_1 \ldots)$, such that for $i = 0, 1, 2, \ldots$ $z_i = x_i + y_i + c_i (mod\ m_i)$, where the numbers c_i are defined inductively by: $c_0 = 0$, and for $i > 0$, $c_i = 1$ if $x_{i-1} + y_{i-1} + c_{i-1} \geq m_{i-1}$ and 0 otherwise. Let us denote by $e = (1, 0, 0, \ldots)$, and define the transformation T on the space $\mathbf{Z}$ by $Tx = x + e$ for $x \in \mathbf{Z}$.

In what follows we shall restrict the choice of the numbers $\eta_{i,j}$ such that if i is an even integer then $\eta_{i,j} = \frac{1}{m_i}$ for all $0 \leq j < m_i$, and if i is an odd integer then $\eta_{i,0} = \eta_i$ for some η_i with $0 < \eta_i < 1$, and $\eta_{i,j} = \frac{1 - \eta_i}{m_i - 1}$ for $0 < j < m_i$. For the purpose of this article we choose the numbers $\eta_{2k+1} > 0$ for $k \geq 0$ such that $\prod_{0 \leq k < \infty} \eta_{2k+1} = P > 0$.

Proposition 1: The transformation T is a 1-1 non-singular transformation defined on the finite measure space $(\mathbf{Z}, \mathcal{B}, \mu)$.

Proof: The fact that T is a 1-1 transformation defined on the measure space $(\mathbf{Z}, \mathcal{B}, \mu)$ is clear. The non-singularity of T follows from the following: For an integer $k \geq 0$ let

$$B_k = \prod_{i=0}^{\infty} Y_i \ \ where \ \ Y_i = \begin{cases} \{m_i - 1\} & \text{if } i < k, \\ \{0, 1, \ldots, m_i - 2\} & \text{if } i = k, \\ \{0, 1, \ldots, m_i - 1\} & \text{if } i > k. \end{cases}$$

Then for each $k \geq 0$, $\mu(TB_k) = \alpha_k \mu(B_{k'})$ for some constant α_k with $0 < \alpha_k \leq 1$. Moreover, T is a linear map on the sets $\{B_k | k \geq 0\}$. A similar argument for T^{-1} shows that $\mu(A) = 0$ if and only if $\mu(TA) = 0$. □

Theorem 1: T is an ergodic transformation defined on the finite measure space $(\mathbf{Z}, \mathcal{B}, \mu)$, and there exists an infinite σ-finite measure m defined on $(\mathbf{Z}, \mathcal{B})$ which is invariant under T and equivalent with the measure μ.

We first prove two Lemmas. In the space $\mathbf{Z}$ let us consider the following subsets: for an even integer $i = 2k \geq 0$ let

$$X_{2k} = \{(x_0, x_1, \ldots) \in \mathbf{Z} \,|\, x_{2j+1} = 0 \ for \ all \ j \geq k\} \quad \text{and} \quad \mathbf{X} = \bigcup_{0 \leq k < \infty} X_{2k}.$$

Lemma 1.1: The subset $\mathbf{X}$ of $\mathbf{Z}$ defined above is an invariant set under the transformation T, and $\mu(\mathbf{X}) = 1$.

Proof: The fact that $\mathbf{X}$ is an invariant set under the transformation T is clear. We recall that $P = \prod_{0 \leq k < \infty} \eta_{2k+1} > 0$, where η_{2k+1} are the numbers defining the measures on the sets $\mathbf{Z}_{2k+1}$ for $k \geq 0$. Then for an even integer $2k > 0$ we have $\mu(X_{2k}) = P/(\prod_{0 \leq j < k} \eta_{2j+1})$. We note that $X_0 \subset X_2 \subset \ldots$ and $\mathbf{X} = \cup_{0 \leq k < \infty} X_{2k}$. This implies that $\mu(\mathbf{X}) = \lim_{k \to \infty} \mu(X_{2k}) = 1$. □

The above says that the measure μ lives on the space $\mathbf{X}$. Therefore, we shall abandon the space $\mathbf{Z}$ and instead consider the non-singular transformation T to be defined on the finite measure space $(\mathbf{X}, \mathcal{B}, \mu)$, where $\mathcal{B}$ and μ are the measurable sets and the measure induced on the set $\mathbf{X}$, respectively.

In $\mathbf{X}$ let us consider the set $X_0 = \{(x_0, x_1, \ldots) \in \mathbf{X} \mid x_{2j+1} = 0 \ for \ all \ j \geq 0\}$ as defined above. It is clear that $\mathbf{X} = \cup_{0 \leq i < \infty} T^i X_0$, and X_0 is a recurrent subset of $\mathbf{X}$ under

T; this means that for any $x \in X_0$ there exists a positive integer n such that $T^n x \in X_0$. Therefore, we define S to be the induced transformation by T on the set X_0; in other words, for $x \in X_0$ let $Sx = T^n x$ where n is the smallest positive integer such that $T^n x \in X_0$.

In the space $(X_0, \mathcal{B}, m)$ we define the following subets: We restrict $i = 2k > 0$ to be an even integer, and let $(\epsilon_0, \epsilon_1, \ldots, \epsilon_{2k-1})$ be a set of $2k$ integers, such that for $0 \leq j < 2k$, $\epsilon_j = 0$ if j is an odd integer, and $0 \leq \epsilon_j < m_j$ otherwise. We let the $2k$-*rectangle* in X_0 be $[\epsilon_0, \epsilon_1, \ldots, \epsilon_{2k-1}] = \{(x_0, x_1, \ldots) \in X_0 \,|\, x_j = \epsilon_j \ for\ 0 \leq j < 2k\}$.

Lemma 1.2: S is an ergodic measure preserving transformation defined on the finite measure space $(X_0, \mathcal{B}, \mu)$.

Proof: We recall that the measure μ on X_0 is a product measure, and for all even integers $i \geq 0$ the measures μ_i were defined such that $\mu_i(p) = \frac{1}{m_i}$ for $p \in \mathbf{Z}_i$. From this follows that S is a measure preserving transformation on $(X_0, \mathcal{B}, \mu)$. To show that S is an ergodic transformation, we consider the $2k$-*rectangles* in X_0. We note that for each integer $k > 0$ there are $P_{2k} = \prod_{1 \leq j < k} m_{2j}$ $2k$-*rectangles* in X_0 each of measure $1/P_{2k}$, and the transformation S maps each one of these $2k$-*rectangles* in X_0 onto another one in a cyclic manner. Furthermore, the set of $2k$-*rectangles* in X_0 for all $k > 0$ is a base for the measurable sets $\mathcal{B}$ of X_0. It follows that if E and F are two sets of positive measure in X_0, then there exists a positive integer n such that $m(T^n E \cap F) > 0$. □

Proof of Theorem 1: Let f be a measurable function defined on $\mathbf{X}$ such that $f(Tx) = f(x)$ for *a.a.* $x \in \mathbf{X}$. It follows that for *a.a.* $x \in X_0$ $f(Sx) = f(x)$. By Lemma 1, since S is an ergodic transformation on $(X_0, \mathcal{B}, \mu)$, we conclude that $f(x) = c$, a constant, for *a.a.* $x \in X_0$. We note that $\mathbf{X} = \cup_{0 \leq i < \infty} T^i X_0$. This implies that $f(x) = c$ for *a.a.* $x \in \mathbf{X}$, and this shows that T is an ergodic transformation defined on the measure space $(\mathbf{X}, \mathcal{B}, \mu)$.

Next we construct the measure m on the space $(\mathbf{X}, \mathcal{B})$ with $m(\mathbf{X}) = \infty$, which will be σ-finite, equivalent with μ and invariant for T.

Let us recall the sets $X_{2k} \subset \mathbf{X}$ for $k \geq 0$, and note that $\mathbf{X} = \cup_{0 \leq k < \infty} X_{2k}$ and $\mu(X_0) = P$, where $P = \prod_{0 \leq k < \infty} \eta_{2k+1}$. We define the measure $m = (1/P)\mu$ on X_0 and extend it to the sets X_{2k} for $k \geq 1$. For $k = 1$ let $B \subset X_2$ be a measurable set; then

$B = \bigcup_{j=0}^{m_1-1} B_j\,(disj)$, where $B_j = \{(x_0, x_1, \ldots) \in B \,|\, x_1 = j\} \quad for \quad 0 \leq j < m_1$. It follows that for $0 < j < m_1$, $\;T^{-j}B_j \subset X_0$, and we define $m(B_j) = m(T^{-j}B_j)$. The measure m then is equivalent with the measure μ on X_2, and $m(X_2) = m_1 m(X_0)$. In general, for each $k > 1$, we let $B \subset X_{2k}$ be a measurable set; then $B = \bigcup_{j=0}^{m_{2k-1}-1} B_j\,(disj)$, where $B_j = \{(x_0, x_1, \ldots) \in B \,|\, x_{2k-1} = j\} \quad for \quad 0 \leq j < m_{2k-1}$. It follows that for $0 < j < m_{2k-1}$, $\;T^{-j}B_j \subset X_0$, and we define $m(B_j) = m(T^{-j}B_j)$. The measure m then is equivalent with the measur μ on X_{2k}, and $m(X_{2k}) = m_{2k-1}m(X_{2k-2})$.

The fact that m is an invariant measure for the transformation T is clear from the construction. □

We shall refer to the ergodic measure preserving transformation T constructed above on the infinite measure measure space $(\mathbf{X}, \mathcal{B}, m)$ as an odometer associated with the sequence $\mathcal{M} = \{m_i\}$. When $m_i \equiv 2$, the transformation T is isomorphic to the one constructed in [7].

Properties of the sequence $\mathcal{M}$ and the transformation T.

The sequence $\mathcal{M} = \{m_i \geq 2\}$, of integers that was used to define the measure space $(\mathbf{X}, \mathcal{B}, m)$ and the odometer T associated with it, possesses some number theoretic properties. The sequence $\mathcal{M}$ also imposes some important geometric restrictions on the transformation T.

Let us define the sequence $\{M_i\}$ by $M_0 = 1$ and $M_i = m_{i-1}M_{i-1}$ for $i > 0$. We let $\mathbf{N} = \{0, 1, 2, \ldots\}$; it follows that every integer $n \in \mathbf{N}$ has a unique representation as a finite sum of integers of the form $n_i M_i$ with $0 \leq n_i < m_i$. In other words, $n = \sum_{0 \leq i < \infty} n_i M_i$, such that for some $k = k(n) > 0$ we have $n_i = 0$ for all $i > k$, and $0 \leq n_i < m_i$ for otherwise. We shall write this representation of $n \in \mathbf{N}$ as $n = (n_0, n_1, \ldots)$, and call it the $\mathcal{M}$-adic representation of n, and refer to n_i as the i-th coordinate of n for $i \geq 0$. We shall denote by $ord_{\mathcal{M}}(n)$ the smallest non-negative integer i for which the i-th coordinate of n is > 0; for the integer 0 we shall say that $ord_{\mathcal{M}}(0) = \infty$.

For a subset $\mathcal{A} \subset \mathbf{N}$ let us denote by $\mathcal{A} - \mathcal{A} = \{n \in \mathbf{N} \,|\, n = a - a' \; for \; a, a' \in \mathcal{A}\}$; By *sums of finite subsets of* $\mathcal{A}$ we mean $SFS(\mathcal{A}) = \{n = \sum_{i \in \mathcal{A}'} i \,|\, \mathcal{A}' \; is\; a\; finite\; subset\; of\; \mathcal{A}\}$.

We consider 0 to be the sum over the empty subset and assume that it belongs to every $SFS(\mathcal{A})$.

In $\mathbf{N}$ let us consider the two subsets $\mathcal{E} = SFS(\bigcup_{k\geq 0}\{m_{2k}\ copies\ of\ M_{2k}\})$ and $\mathcal{F} = SFS(\bigcup_{k\geq 0}\{m_{2k+1}\ copies\ of\ M_{2k+1}\})$. It follows that $\mathcal{E} \oplus \mathcal{F} = \mathbf{N}$, where by $\mathcal{E} \oplus \mathcal{F}$ we mean $\{e + f \,|\, e \in \mathcal{E},\ f \in \mathcal{F}\}$ such that if $e + f = e' + f'$ then $e = e'$ and $f = f'$; see [3]. We note that $ord_{\mathcal{M}}(n)$ is an even integer for every $n \in \mathcal{E} - \mathcal{E}$, and $ord_{\mathcal{M}}(n)$ is an odd integer for every $n \in \mathcal{F} - \mathcal{F}$. Thus, the sequences $\mathcal{E}$ and $\mathcal{F}$ (the odd and even parts of $\mathcal{M}$) exhibit complementing or dual properties as subsequences of $\mathbf{N}$. In a similar way, we shall show that these sequences exhibit important geometric (wandering and recurrence) properties of the transformation T.

Theorem 2: Let $(\mathbf{X}, \mathcal{B}, m)$ be the σ-finite measure space and T the ergodic measure preserving odometer associated with the sequence $\mathcal{M} = \{m_i\}$. Then for the subset $X_0 \subset X$ and the set of integers $\mathcal{F}$ described above we have $\mathbf{X} = \bigcup_{f\in\mathcal{F}} T^f X_0\,(disj)$.

Proof: Let us recall the set $X_0 = \{(x_0, x_1, \ldots) \in X \,|\, x_{2j+1} = 0\ for\ all\ j \geq 0\}$, the sequence of integers $\{M_i\}$ defined by $M_0 = 1$ and $M_{i+1} = m_i M_i$ for $i \geq 0$, and the sequence $\mathcal{F} = SFS(\bigcup_{i\geq 0}\{m_{2k+1}\ copies\ of\ M_{2k+1}\})$. Then in the $\mathcal{M}$-adic representation of an integer $n \in \mathbf{N}$, we note that for $i \geq 0$, $M_i = (\underbrace{0, \ldots, 0}_{i-zeroes}, 1, 0, \ldots)$. In other words, the $\mathcal{M}$-adic representation of the integer M_i has a 1 in its i-th coordinate and 0 elsewhere. It follows that for $k \geq 0$ we have $X_{2k+2} = \bigcup_{0\leq j<m_{2k+1}} T^{jM_{2k+1}} X_{2k}\ (disj)$. Since $X = \bigcup_{0\leq k<\infty} X_{2k}$, it follows that $X = \bigcup_{f\in\mathcal{F}} T^f X_0\ (disj)$. $\square$

Another way stating the conclusion of the above Theorem 2 is: the set X_0 is an exhausting weakly wandering set for the odometer transformation T under the sequence $\mathcal{F}$. We note that the sequence $\mathcal{F}$ is related to the 'odd-indexed' subsequence of $\mathcal{M}$. In the next section we consider the 'even-indexed' subsequence of $\mathcal{M}$.

The sequence M and multiple recurrence.

Definition: Let $p > 0$ be a positive integer. The transformation T defined on the measure space $(\mathbf{X}, \mathcal{B}, m)$ is said to be *p-recurrent* if for any measurable set $B \in \mathcal{B}$ with $m(B) > 0$

there exists a positive integer $n > 0$ such that $m(B \cap T^n B \ldots \cap T^{(p-1)n} B) > 0$. The transformation T is said to be *multiply recurrent* if T is p-recurrent for every $p > 0$.

We need to consider $2k$-*rectangles* in the space $\mathbf{X}$. Again we restrict $i = 2k > 0$ to be an even integer, and let $(\epsilon_0, \epsilon_1, \ldots, \epsilon_{2k-1})$ be a set of $2k$ integers, such that $0 \leq \epsilon_j < m_j$ for $0 \leq j < 2k$. Then a $2k$-*rectangle* in $\mathbf{X}$ is a subset of X_{2k} and is $[\epsilon_0, \epsilon_1, \ldots, \epsilon_{2k-1}] = \{(x_0, x_1, \ldots) \in X_{2k} \,|\, x_j = \epsilon_j \; for \; 0 \leq j < 2k\}$. It follows that the $2k$-*rectangles* in $\mathbf{X}$ approximate the subsets of $\mathbf{X}$ of finite measure.

Theorem 3: Let $(\mathbf{X}, \mathcal{B}, m)$ be the σ-finite measure space and T the ergodic measure preserving odometer associated with the sequence $\mathcal{M} = \{m_i\}$. Then
i) $\overline{\lim_{i \to \infty}} m_{2i+1} = \infty$ if and only if T is multiply recurrent.
i) If $p = \overline{\lim_{i \to \infty}} m_{2i+1} < \infty$, then T is p-recurrent but not $(p+1)$-recurrent.

Lemma 3.1: Let $i = 2k \geq 0$ be an even integer, and let R be the union of $2k$-*rectangles* in $\mathbf{X}$. Then for any integer j with $0 < j < m_i$

$$m(R \cap T^{M_i} R \cap \ldots \cap T^{jM_i} R) = (1 - j/m_i) m(R).$$

Proof: Let $i = 2k > 0$ be an even integer, and let R be the union of $2k$-*rectangles* in $\mathbf{X}$. Then any $2k$-*rectangle* $[\epsilon_0, \ldots, \epsilon_{2k-1}]$ in $\mathbf{X}$ splits into m_{2k} $(2k+2)$-*rectangles*; namely, $[\epsilon_0, \ldots, \epsilon_{2k-1}, \eta, 0]$ for $0 \leq \eta < m_{2k}$. Each one of these $(2k+2)$-*rectangles* has measure $(1/m_{2k}) m([\epsilon_0, \ldots, \epsilon_{2k-1}])$, and

$$T^{M_{2k}} [\epsilon_0, \ldots, \epsilon_{2k-1}, \eta, 0] = [\epsilon_0, \ldots, \epsilon_{2k-1}, \eta + 1, 0]$$

for $0 \leq \eta < m_{2k} - 1$. Furthermore, the set

$$T^{M_{2k}} [\epsilon_0, \ldots, \epsilon_{2k-1}, m_{2k} - 1, 0] = [\epsilon_0, \ldots, \epsilon_{2k-1}, 0, 1]$$

is disjoint from any $2k$-*rectangle* in $\mathbf{X}$. □

Lemma 3.2: Let $x \in X_0$, and let $n > 0$ be a positive integer with $ord_{\mathcal{M}}(n) = r$. Then there exists an integer j with $1 \leq j \leq m_r$ such that $T^{jn} x \notin X_0$.

Proof: Let $x \in X_0,\ n > 0$, and $ord_{\mathcal{M}}(n) = r$. In the $\mathcal{M}$-adic representation of the integer n let us denote by p the r-th coordinate of n. We note that $0 < p < m_r$, and for all $i < r$ the i-th coordinate of n is 0.

Suppose $ord_{\mathcal{M}}(n) = r$ is an odd integer; then the r-th coordinate of $T^n x$ will equal to $p > 0$. This says that $T^n x \notin X_0$. Thus $j = 1$ in this case.

Therefore, we assume that $ord_{\mathcal{M}}(n) = r$ is an even integer. Let t equal the r-th coordinate of the point $x \in X_0$. It is clear that $0 \leq t < m_r$, and the $(r+1)$-th coordinate of x equals 0. Let $q =$ the $(r+1)$-th coordinate of n; it follows that $0 \leq q < m_{r+1}$.

If $0 < q < m_{r+1} - 1$, then since the $(r+1)$-th coordinate of $T^n x$ is > 0, it follows that $T^n x \notin X_0$, and thus $j = 1$.

Next we let $q = 0$ and let j be the smallest positive integer such that $t + jp \geq m_r$. It follows that $T^{jn} \notin X_0$, and $1 \leq j \leq m_r$.

Finally, we let $q = m_{r+1} - 1$, $t_1 = t$, and note that $0 \leq t_1 < m_r$. It follows that if $t_1 + p < m_r$ then $T^{jn} x \notin X_0$ for $j = 1$. Otherwise, we let $t_2 = t_1 + p - m_r$ and note that $0 \leq t_2 < t_1$. It follows that if $t_2 + p < m_r$ then $T^{jn} x \notin X_0$ for $j = 2$. Otherwise, we let $t_3 = t_2 + p - m_r$ and note that $0 \leq t_3 < t_2$. We continue this way for at most m_r steps and stop when $t_j + p < m_r$ for some j with $1 \leq j \leq m_r$. $\square$

For the proof of Theorem 3 we need the following Corollary. It is a sharpened version of Lemma 3.2, and is proven in a similar way. We omit its proof.

Corollary: For some integer $q > 0$ let $V = \{(x_0, x_1, \ldots) \in X_0 \,|\, x_i = 0 \ for\ i \leq q\}$. Let $x \in V$, and let $n > 0$ be a positive integer with $ord_{\mathcal{M}}(n) = r$. Then there exists an integer j with $1 \leq j \leq m_r$ such that $T^{jn} x \notin V$.

Proof of Theorem 3: Suppose $\overline{\lim\limits_{i \to \infty}} m_{2i+1} = \infty$, and let $B \in \mathcal{B}$ be a set of positive measure. Then for any integer $p > 0$ there exist arbitrarily large even integers $i = 2k$ such that $1 - p/m_i \geq 1/p$. By possibly considering a subset of B, we may assume that $0 < m(B) < \infty$. Since the $2k$-*rectangles* approximate the sets of finite measure, it follows that there exists an even integer $i = 2k > 0$ and a set R, which is the union of $2k$-*rectangles* in $\mathbf{X}$, such that $m(R\Delta B) < (1/p)m(R)$. Lemma 3.1 then implies

$$m(R \cap T^{M_i}R \cap \ldots \cap T^{(p-1)M_i}R) = (1 - p/m_i)m(R) > (1/p)m(R).$$

It follows that $m(B \cap T^{M_i}B \cap \ldots \cap T^{(p-1)M_i}B) > 0$, and this shows that T is p-recurrent for any $p > 0$.

Next we let $p = \overline{\lim_{i\to\infty}} m_{2i} < \infty$. Then there exists an integer $q > 0$, and we assume it to be even, such that $m_i \leq p$ for all $i > q$. From the Corollary to Lemma 3.2 follows that if $V = \{(x_0, x_1, \ldots) \in X_0 \,|\, x_i = 0 \;\mathit{for}\; 0 \leq i \leq q\}$, then there exists an integer j with $1 \leq j \leq p$ such that $T^{jn}x \notin V$. This implies that $V \cap T^n V \cap \ldots \cap T^{pn}V = \emptyset$, which say that the transformation T is not $(p+1)$-recurrent. In particular, T is not multiply recurrent.

Finally we show that the transformation T is p-recurrent in this case. Since $p = \overline{\lim_{i\to\infty}} m_{2i+1} < \infty$ it follows that there exist arbitrarily large even integers $i = 2k > 0$ such that $m_i = p$. Now let $B \in \mathcal{B}$ be a set of positive measure. Again we may assume that $0 < m(B) < \infty$, and choose a large even integer $i = 2k > 0$ and a set R, which is the union of $2k$-rectangles in $\mathbf{X}$, such that $m(R\Delta B) < (1/p)m(R)$. We let $M_i = \prod_{1\leq j\leq i} m_j$; Lemma 3.1 then implies $m(R \cap T^{M_i}R \cap \ldots \cap T^{(p-1)M_i}R) = (1/p)m(R)$. It follows that $m(B \cap T^{M_i}B \cap \ldots \cap T^{(p-1)M_i}B) > 0$, and this shows that T is p-recurrent. $\square$.

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