Universally bad Integers and the 2-adics

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Good Pairs of Integers

$\mathbb{S} = \{0, 1, 4, 5, 16, 17, 20, 21, \cdots\}$

i.e., sums of finite subsets of the

even powers of 2.

Fact: $\mathbb{Z} = \mathbb{S} \ominus 2\mathbb{S}$

That is, each integer is obtained, and it is obtained uniquely.

- $\forall n \in \mathbb{Z}, \exists s_i \in \mathbb{S}$ such that $n = s_1 - 2s_2$
- $s_1 - 2s_2 = t_1 - 2t_2 \Rightarrow s_1 = t_1$ and $s_2 = t_2$

DEF. (de Bruijn, 1950, 1964) A pair (a, b) of positive odd integers is **good** if

$$\mathbb{Z} = a\mathbb{S} \ominus 2b\mathbb{S}$$

otherwise the pair (a, b) is **bad**

Digression to Ergodic Theory

In 1970, Hajian and Kakutani presented an example of an Ergodic Infinite Measure Preserving Transformation with an Exhaustive Weakly Wandering.

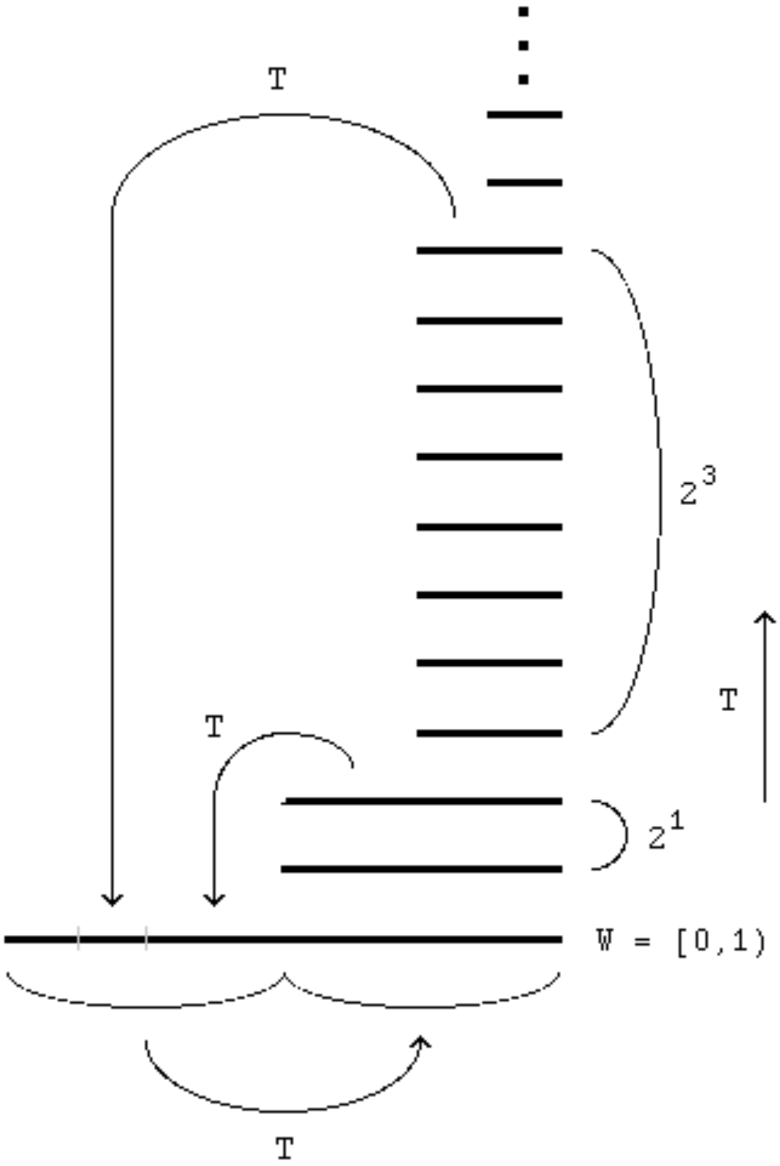
DEF: $\mathbb{A}$ a sequence of integers is **Exhaustive Weakly Wandering** for the transformation T if there exists a set W of positive measure satisfying

1: $\mu(T^aW \cap T^{a'}W) = 0$, $a \neq a'$, $a, a' \in \mathbb{A}$.

2: $\mu(X \setminus \cup_{a\in\mathbb{A}} T^aW) = 0$.

The set W is then an **Exhaustive Weakly Wandering Set** for T under $\mathbb{A}$.

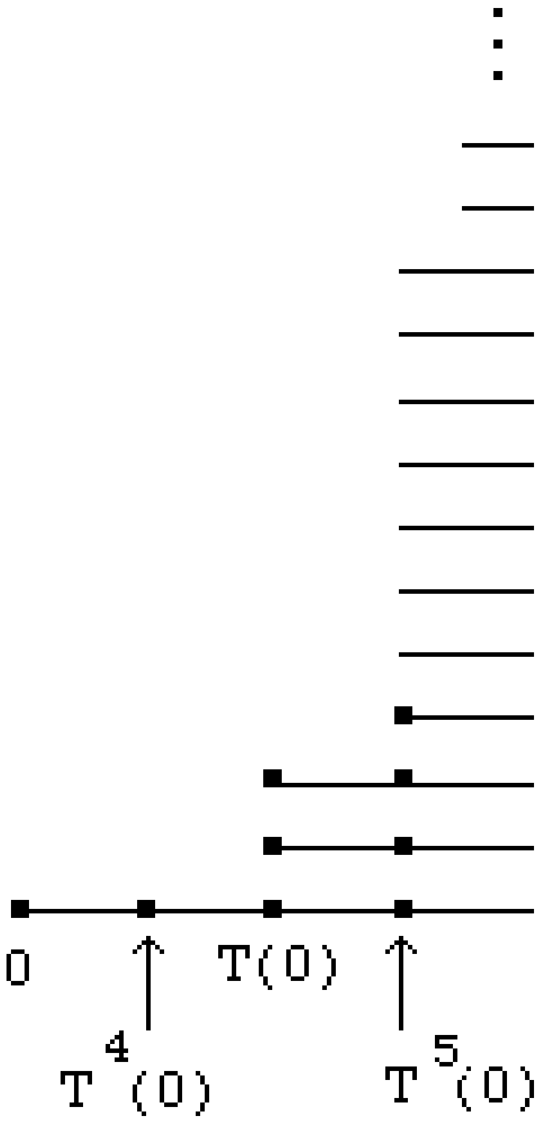
The example is a skyscraper construction over the unit interval.


T
T
2^3
T
2^1
W = [0,1)
T

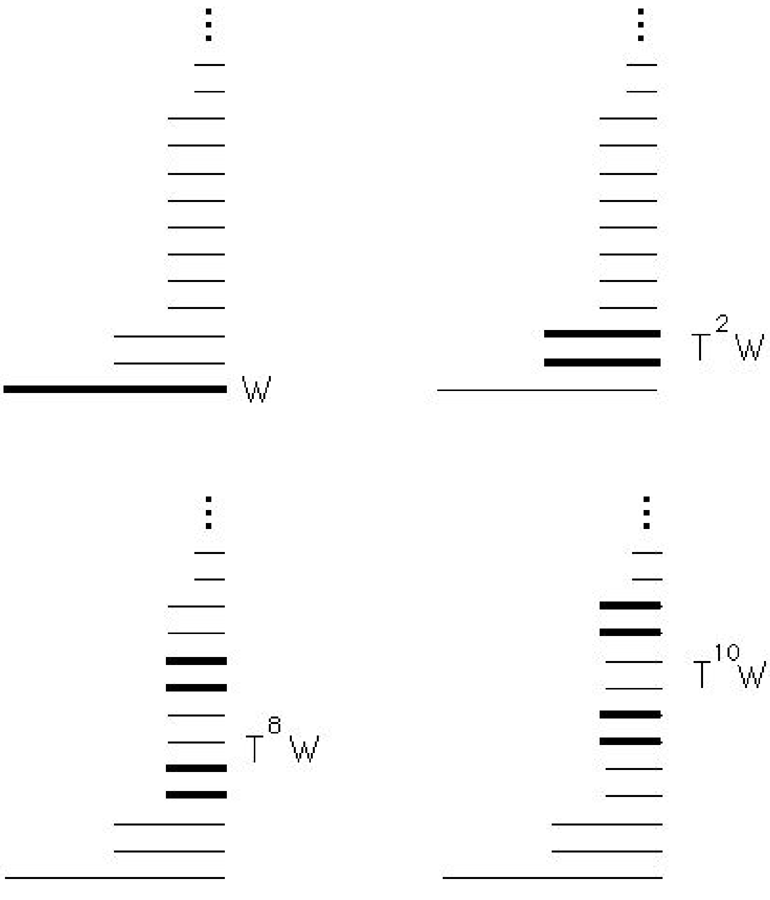
Consider the point $0 \in [0,1) = W$.

It's return sequence

$$H(0) = \{n \ : \ T^n(0) \in W\} = \mathbb{S}.$$



While the exhaustive weakly wandering sequence for the set $W = [0,1)$ is $2\mathbb{S}$.



A similar transformation and skyscraper construction corresponds to any good pair of integers.

Back to Good Pairs

$\mathbb{S} = \{0, 1, 4, 5, 16, 17, 20, 21, \cdots\}$

(a, b) positive odd integers is **good** if $\mathbb{Z} = a\mathbb{S} \ominus 2b\mathbb{S}$. Otherwise the pair is called **Bad**.

Examples: $(1, 1)$, $(1, 7)$, and $(7, 13)$.

Example: $(1, 2^{2k+1} - 1)$ good for all $k \geq 0$.

Quick Facts from de Bruijn

- (a, b) good requires $\gcd(a, b) = 1$
- (a, b) good iff (b, a) good.
- (a, b) good requires $a \equiv b$ Mod 6

de Bruijn (1950) listed good pairs (a, b) for $1 \leq a \leq b \leq 100$ obtained by “pencil and paper” and “shuffling four strips of paper”

de Bruijn (1964) listed all good pairs up to 1800, calculated by computer.

DEF: A positive odd integer u is **Universally Bad (U.B.)** if (ua, b) is bad for all pairs of positive odd integers a and b.

THM. (De Bruijn) $u = 2^k + 1$ are U.B.

THM. $u = \phi_{p^k}(4)$ are U.B., where p is prime and $\phi_n(x)$ is the n'th cyclotomic polynomial.

The first two U.B. integers are 3 And 5.

- We will see they are bad in slightly different ways.

Using this difference we will define de Bruijn U.B. integers.

THM 3, $2^{2k+1} + 1$, and $\phi_{p^k}(4)$, $p > 2$ prime will all be de Bruijn U.B.

THM 5, and $2^{2k} + 1$ will be U.B. but not de Bruijn U.B.

The analysis is via the 2-adics.

2-Adics $\mathbb{Z}_2$

The 2-adic integers is the completion of the nonnegative integers in the 2-adic norm.

$$\mathbb{Z}_2 = \left\{ z = \sum_{i \geq 0} z_i 2^i \ : \ z_i \in \{0, 1\} \right\}$$

Write $0 < n = 2^k \cdot m$, m odd.

DEF. The 2-adic **order** of $n > 0$ is defined by $ord(n) = ord_2(n) = k$, if k is the highest power of 2 which divides n.

DEF. The 2-adic **norm** is $|n| = |n|_2 = 2^{-ord(n)} = 2^{-k}$.

By convention: $ord(0) = \infty$ and $|0| = 0$.

Identify $\mathbb{Z}_2$ with $\{0,1\}^{\mathbb{N}}$,

i.e. $z = \sum z_i 2^i \leftrightarrow (z_0, z_1, z_2, \cdots)$.

Extend ord to $\mathbb{Z}_2$ by setting

$ord(z) = i$, coordinate of first nonzero z_i

where $z = \sum z_i 2^i = (z_0 z_1 \cdots)$ for $z \in \mathbb{Z}_2$.

Extend the 2-adic norm $|z| = 2^{-ord(z)}$.

Thus we get all integers represented in $\mathbb{Z}_2$

Positive integers end in all 0's

Negative integers end in all 1's

Addition in this representation is coordinate-wise from left to right with "carry" to the right.

- We can speak of convergence using the norm or the ord.

- A sequence $z(n) \in \mathbb{Z}_2$ converges if and only if

$$ord(z(n) - z(m)) \to \infty \text{ as } n, m \to \infty.$$

Lemma (**Geometric Series**) If $ord(x) > 1$,

(*i.e.* $|x|_2 < 1$) then

$$\sum_0^\infty x^i = \frac{1}{1-x}$$

in the 2-adic integers.

Illustrations

$-1 = (\bar{1}) = \sum_0^\infty 2^i = \frac{1}{1-2}$.

Repeating pattern $(\overline{10}) = \sum_0^\infty 4^i = \frac{1}{1-4} = -\frac{1}{3}$,

and in general, $\sum_{i=0}^\infty 4^{in} = \frac{1}{1-4^n}$.

Note that these latter two are 2-adic integers in the closure of $\mathbb{S}$.

$\overline{\mathbb{S}}$ denotes the closure of $\mathbb{S}$ in the 2-adic norm.

Fact: $\mathbb{S}$ and its closure $\overline{\mathbb{S}}$ have 0's in all odd coordinate places; i.e., $\alpha = (\alpha_0 \alpha_1 \cdots) \in \overline{\mathbb{S}}$ if and only if $\alpha_{2i+1} = 0$ for all i.

Hence, as on previous page $-1/3 = (\overline{10}) \in \overline{\mathbb{S}}$.

Fundamental Theorem: For all pairs (a, b) of positive odd integers, the 2-adic integers can be written as

$$\mathbb{Z}_2 = \overline{a\mathbb{S} \ominus 2b\mathbb{S}} = \overline{a\mathbb{S}} \ominus \overline{2b\mathbb{S}}.$$

Idea. The Fundamental Theorem clarifies how it is possible for $\mathbb{Z} \neq a\mathbb{S} \ominus 2b\mathbb{S}$.

THM. The pair of positive odd integers (a, b) is bad, *i.e.* $\mathbb{Z} \neq a\mathbb{S} \ominus 2b\mathbb{S}$, if and only if there is an integer n such that

$$n = a\sigma - 2b\tau$$

where σ or $\tau \in \overline{\mathbb{S}} \backslash \mathbb{S}$.

Because then this integer could not be obtained in a second manner.

Illustration of why 3 is Universally Bad

Let (a, b) be any pair of odd positive integers; we need to show that $(3a, b)$ is bad.

The fraction $\frac{1}{3} = (\overline{10})$ belongs to $\overline{\mathbb{S}} \setminus \mathbb{S}$.

Putting $\sigma = -\frac{1}{3}$ and $\tau = 0$

we have $-a = 3a\sigma - 2b \cdot 0$ and so $(3a, b)$ is bad.

DEF. An odd positive integer u is a **de Bruijn universally bad integer** if there is some $\sigma \in \overline{\mathbb{S}} \setminus \mathbb{S}$ such that $u\sigma \in \mathbb{Z}$.

Example.

The integer $u = 85$ is de Bruijn universally bad.

To see this, observe that the fraction

$$-21/255 = (\overline{10101000}) = \frac{1+4+4^2}{1-4^4} \in \overline{\mathbb{S}} \setminus \mathbb{S},$$

and $85 \cdot \frac{-21}{255} = -7$.

Example

The integer $u = 341$ is de Bruijn universally bad.

In this case, the fraction

$$-\frac{81}{1023} = (\overline{1000101000}) = \frac{1+4^2+4^3}{1-4^5}$$

is in $\overline{\mathbb{S}} \setminus \mathbb{S}$ and $341 \cdot \frac{-81}{1023} = -27$.

The integer 341 is a new universally bad integer not on de Bruijn's list. It is a product of 11 and 31 neither of which is universally bad.

The integer 85 is on de Bruijn's list since it is a multiple of $5 = 2^2 + 1$ and $17 = 2^4 + 1$, which are universally bad, though neither is a de Bruijn universally bad integer.

These illustrate

THM. A positive odd integer u is a de Bruijn universally bad integer if and only if there exists a fraction of the form

$$\sigma = \frac{\sum_{i=0}^{R-1} \delta_i 4^i}{1 - 4^R} \in \overline{\mathbb{S}} \setminus \mathbb{S}$$

with $\delta_i \in \{0, 1\}$, and such that $u\sigma \in \mathbb{Z}$.

Illustration of why 5 is Universally Bad

5 is not de Bruijn universally bad . But it is still Universally Bad.

The two numbers

$$-\frac{1}{3} \text{ and } -\frac{1}{15} = (1000\overline{1000}) = \sum 4^{2i}$$

are both in $\overline{\mathbb{S}} \setminus \mathbb{S}$.

Multiply by 5: $-\frac{5}{3}$ and $-\frac{1}{3}$ are both in $5(\overline{\mathbb{S}} \setminus \mathbb{S})$.

Now multiplying by a (assuming it is not a multiple of 3) the set of fractional parts of the two numbers $-\frac{a}{3}$ and $-\frac{5a}{3}$ will be $\{1/3, 2/3\}$.

On the other hand, since $\tau = -1/3 \in \overline{\mathbb{S}} \setminus \mathbb{S}$ it follows that $-2b\tau$ has a fractional part of 1/3 or 2/3.

Choose $\sigma \in 5a(\overline{\mathbb{S}} \setminus \mathbb{S})$ to be either $-\frac{a}{3}$ or $-\frac{5a}{3}$, so that $\sigma - 2b\tau \in \mathbb{Z}$.

QED

Back to Proof of Fund. Thm

Fundamental Theorem: For all pairs (a, b) of positive odd integers, the 2-adic integers can be written as

$$\mathbb{Z}_2 = \overline{a\mathbb{S} \ominus 2b\mathbb{S}} = \overline{a\mathbb{S}} \ominus \overline{2b\mathbb{S}}.$$

This is not true for arbitrary subsets of $\mathbb{Z}_2$.

Def A 2-adic integer $z \in \mathbb{Z}_2$ is of **even order** if $ord(z) = 2i$ and of **odd order** if $ord(z) = 2i+1$.

This leads to slight Language Issue

Language Issue: All odd integers a, (positive and negative) are of even order: in fact, $ord(a) = 0$ since the highest power of 2 which divides an odd integer a is 2^0.

Even integers may be either odd order or even order. By convention, 0 is considered both even order and odd order, and is the only such number.

It is easy to see

Lemma. Multiplication by an odd integer is ord-preserving.

Actually multiplication by $z \in \mathbb{Z}_2$ where $ord(z) = 0$ is ord-preserving because it just leaves the first non-zero coordinate in the same location.

The theorem is proved through a series of elementary lemmas (*which extends to p-adics*).

Lemmas

(a) s is of even order for all $s \in \mathbb{S}$.

(b) $s - s'$ is of even order for all $s, s' \in \mathbb{S}$

Let a be a positive odd integer. Then,

(c) $a\sigma$ is of even order for all $\sigma \in \overline{\mathbb{S}}$

and $ord(a\sigma) = ord(\sigma)$.

(d) $a\sigma - a\sigma'$ is of even order for all $\sigma, \sigma' \in \overline{\mathbb{S}}$

and $ord(a\sigma - a\sigma') = ord(\sigma - \sigma')$.

Let $\tau, \tau' \in \overline{2\mathbb{S}}$ and let b be an odd integer. Then,

(e) $ord(b\tau) = ord(\tau)$

and each element in $\overline{2\mathbb{S}}$ is of odd order.

(f) $ord(b\tau - b\tau') = ord(\tau - \tau')$

and each element in $\overline{2\mathbb{S}} - \overline{2\mathbb{S}}$ is of odd order.

Some Proofs

Proof: of (1) $\overline{a\mathbb{S}} - \overline{2b\mathbb{S}} = \overline{a\mathbb{S}} \ominus \overline{2b\mathbb{S}}$

We want to show that each difference is unique.

Suppose $a\sigma - 2b\tau = a\sigma' - 2b\tau'$.

Rearranging the terms gives

$$a(\sigma - \sigma') = b(2\tau - 2\tau').$$

The left hand side has even order and the right hand side has odd order. The only number with both odd and even order is 0.

QED

Proof: of (2) $\overline{a\mathbb{S} - 2b\mathbb{S}} = a\overline{\mathbb{S}} - 2b\overline{\mathbb{S}}$.

One containment, $\overline{a\mathbb{S} - 2b\mathbb{S}} \supset a\overline{\mathbb{S}} - 2b\overline{\mathbb{S}}$, is obvious.

The other containment follows from

Lemma. Suppose $as(n) - 2bt(n) \to z$ with sequences $s(n), t(n) \in \mathbb{S}$ (and a, b are positive odd integers). Then $s(n)$ and $t(n)$ each converge.

The latter follows by rewriting the Cauchy difference

$$as(n) - 2bt(n) - (as(m) - 2bt(m))$$

$$= [as(n) - as(m)] - b[2(t(n) - 2(t(m))]$$

Now us the fact that when $ord(x) \neq ord(y)$ then

$$ord(x \pm y) = min(ord(x), ord(y)).$$

Apply this to $x = [as(n) - as(m)]$ which has even order and $y = b[2(t(n) - 2(t(m))]$ which has odd order.

Some Questions

Question If u is a de Bruijn universally bad integer, is it divisible by 3 or $\phi_{p^k}(4)$ for some p and k?

Question The integers $4^k + 1$ (and their multiples) are universally bad but not de Bruijn universally bad. What other integers are universally bad but not de Bruijn universally bad?

Question What is the (upper and lower) density of the universally bad integers? Assuming the density exists, it is certainly more than 1/3.

Question Given an a which has at least one b with which it is a good pair, are there infinitely many b's with which it is a good pair? de Bruijn shows that this is true for $a = 1$.

Question Is there an algorithm — meaning some program that stops in a finite number of steps — which determines if a given integer is universally bad?

References

N. G. de Bruijn, On bases for the set of integers, *Publ. Math. Debrecen* **1** (1950), 232–242.

N. G. de Bruijn, Some direct decompositions of the set of integers, *Math. Comp.* **18** (1964), 537–546.

Presentation based upon: *Universally Bad Integers and the 2-Adics*, S. Eigen, Y. Ito and V.S. Prasad. J. Num. Theory 107 (2004), pg 322-334.

Proof: **of (3)** $\mathbb{Z}_2 = \overline{a\mathbb{S} - 2b\mathbb{S}}$.

It is enough to show that $a\mathbb{S} - 2b\mathbb{S}$ is dense in $\mathbb{Z}$ which is in turn dense in $\mathbb{Z}_2$.

The proof is technically messy - though the idea is clear.