Infinite Measure Preserving Transformations,

Complementing Subsets of the Integers

and

Compact Abelian Groups

A major difference between finite and infinite measure preserving maps is that there always exist weakly wandering sequences in the infinite measure case (and there never exist such sequences in the finite measure preserving case).

T an ergodic, infinite measure preserving, invertible tranformation on a non-atomic measure space $(X, \mathcal{B}, \mu)$.

$\mathbb{A}$ a sequence of integers is **Exhaustive Weakly Wandering** for T if there exists a set W satisfying

1: $\mu(T^a W \cap T^{a'} W) = 0$, $a \neq a'$, $a, a' \in \mathbb{A}$.

2: $\mu(X \setminus \cup_{a \in \mathbb{A}} T^a W) = 0$.

The set W is then an **Exhaustive Weakly Wandering Set** for T under $\mathbb{A}$.

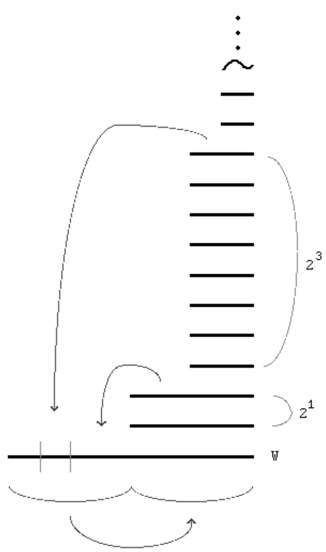
Example: Hajian-Kakutani

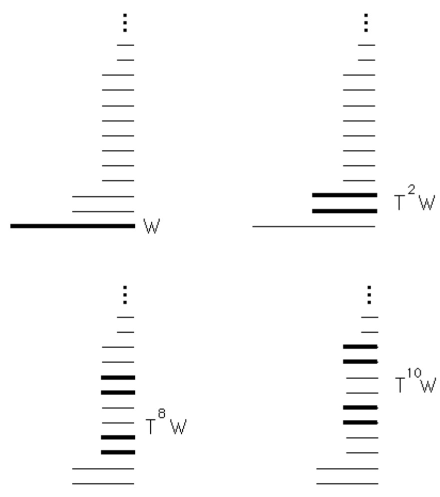
The first known example is due to Hajian and Kakutani.

The set $W = [0, 1]$ and the space X is a skyscraper above W. The induced map T_W is the Von Neumann map *a.k.a.* the adding-machine, *a.k.a.* the odometer map.

The exhaustive weakly wandering sequence is

$\mathbb{A} = \{0, 2, 8, 10, 32, 34, \cdots\} = SFS(2^{2k+1}, k \geq 0)$, *i.e.* finite sums of odd powers of 2.

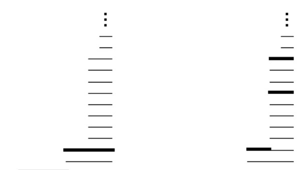




Questions

Some of the questions one asks are

Given T and $\mathbb{A}$ find all the exhaustive weakly wandering sets.



Given T and W find all the exhaustive weakly wandering sequences.

Given T find all the exhaustive weakly wandering sequences and sets.

Given $\mathbb{A}$ find all the T which have $\mathbb{A}$ as a sequence. (In this case, one usually adds the condition that images $T^i W$ generate the measurable sets.)

Complementing Subsets of $\mathbb{Z}$

An exhaustive weakly wandering sequence always has a complement in the integers $\mathbb{Z}$.

Two infinite sequences of integers $\mathbb{C}$ and $\mathbb{D}$ are a **Complementing Pair of Subsets of** $\mathbb{Z}$ if every integer n can be obtained as a unique sum $n = c+d$ with $c \in \mathbb{C}$ and $d \in \mathbb{D}$.

Denote this $\mathbb{C} \oplus \mathbb{D} = \mathbb{Z}$.

Example: $\mathbb{A} = SFS\{2^{odd}\}$ $\mathbb{B} = SFS\{2^{even}\}$

$$\mathbb{A} \oplus \Big(-\mathbb{B} \Big) = \mathbb{Z}$$

Note, $\mathbb{A} \oplus \mathbb{B} = \mathbb{N}$

For a particular exhaustive weakly wandering sequence $\mathbb{C}$, we may construct complements from the hitting sequence of a "typical" point.

Let $w \in W$.

The **Hitting Sequence** $\mathbb{H}(w) = \{n \in \mathbb{Z} : T^n(w) \in W\}$.

Then for almost all points w, $\mathbb{C} \oplus \mathbb{H}(w) = \mathbb{Z}$.

The previous discussion makes it appear that the two members of the pair $\mathbb{C}, \mathbb{H}(w)$ are quite different. However, in many examples they are the same "type" of sequence.

To illustrate this, we will re-present the earlier Hajian-Kakutani example in the framework of a compact abelian group.

Note that the Hajian-Kakutani example is based on powers of 2 and this puts us into a specific compact abelian group.

Questions

Some of the questions one asks are

Given a sequence of integers, does it have a complement.

Given a sequence of integers, how does one find a complement if it exists.

Characterize the complements of a given sequence (which is known to have at least one complement).

A common theme that runs through these questions, is that "finite changes" often result in new complements, but the limiting cases don't.

Illustration: The sequence $\mathbb{A} = SFS\{2^{odd}\}$ has the complement $-\mathbb{B} = SFS\{2^{even}\}$.

We can modify $\mathbb{B}$ by replacing 2^0 with $2^0 \cdot 3$ and then take finite sums. This gives another complement of $\mathbb{A}$.

Likewise, any finite number of 2^{2k} may be replaced with $2^{2k} \cdot 3$ (or any odd integer) and obtain another complement. However, if all the terms are multiplied by 3, we no longer have a complement.

Referring back to the "other exhaustive weakly wandering sets" given on page 6, The first corresponds to replacing 2^0 with 3, and the second corresponds to replacing $2^0, 2^2$ with $3, 12$. (This is for $+\mathbb{B}$.)

The 2−adic Integers: $\mathbb{Z}_2$

Essentially, the 2−adic Integers are infinite sequences of 0 and 1's with the usual topology and product measure $\prod\{\frac{1}{2}, \frac{1}{2}\}$.

Addition however is Mod 2 with carry to the right.

$\mathbb{Z}_2 = \{\alpha = \sum_{i=0}^{\infty} \alpha_i 2^i\}$ where $\alpha_i = 0$ or 1.

The α_i are the 2−**adic digits** of $\alpha = (\alpha_i)$.

Addition (and multiplication) are defined in the obvious manner.

The positive integers are given by finite sums; the negative integers are given by sums with only a finite number of non-zero 2−adic digits.

The 2−**adic valuation** is given by $|\alpha|_2 = 2^{-k}$ where α_k is the first non-zero 2−adic digit in α. (Note, $|0|_2 = 0$.)

This gives a metric $|\alpha-\beta|_2$ in which all the infinite sums α converge.

A basis for the topology is given by the open balls of radius 2^{-k}

$$\mathcal{B}_k(\alpha) = \{\beta : \alpha = \beta (mod 2^k)\}$$

The 2−adic (Haar) measure μ_2 satisfies

$$\mu_2\Big(\mathcal{B}_k(\alpha)\Big) = 2^{-k}$$

T We will be primarily concerned with the map $T : \mathbb{Z}_2 \to \mathbb{Z}_2$ corresponding to adding 1, *i.e.* $T(\alpha) = \alpha + 1$.

Reconstuction of Hajian-Kakutani Example

Consider the sequence $\mathbb{B} = \{0, 1, 4, 5, 16, 17, \cdots\} = SFS(2^{2k}, k \geq 0)$, *i.e.* finite sums of even powers of 2.

Take the closure $\bar{\mathbb{B}}$ in $\mathbb{Z}_2$.

This is a set of measure zero.

However, the closure $\bar{\mathbb{A}}$ of $\mathbb{A} = \{0, 2, 8, 10 \cdots\} = SFS(2^{2k+1}, k \geq 0)$, *i.e.* finite sums of odd powers of 2 satisfies

$$\bar{\mathbb{A}} \oplus \bar{\mathbb{B}} = \mathbb{Z}_2$$

Using this, we can project the Haar measure μ_2 onto a probability measure $\mu_{\mathbb{B}}$ on $\bar{\mathbb{B}}$.

Note, modulo some additional conditions, the measure on $\bar{\mathbb{B}}$ is independent of its complement in $\mathbb{Z}_2$. Thus in a certain sense the projected measure on $\bar{\mathbb{B}}$ is ”natural”.

Define $X = X_{\mathbb{B}} = \cup_{i=-\infty}^{\infty} T^i \bar{\mathbb{B}}$, and extend the measure $\mu_{\mathbb{B}}$ up the skyscraper in the obvious manner to an infinite measure μ_X.

$(X_{\mathbb{B}}, T, \mu_X)$ ”is” the Hajian-Kakutani example.

To clarify this:

First remove from X the orbit of 0.

$\beta = (\beta_i) \in \bar{\mathbb{B}}$ has $\beta_{2i+1} = 0, \forall i \geq 0$.

When we ”add” T^a, $a \in \mathbb{A}$, we are putting a ”1” in an odd digit thus obtaining a disjoint set.

Some Properties to observe:

$|a - a'|_2 = 2^{-2k-1}$ for some k, $a \neq a' \in \mathbb{A}$.

$|b - b'|_2 = 2^{-2k}$ for some k, $b \neq b' \in \mathbb{B}$.

Both sets are **maximal** in $\mathbb{Z}$ with respect to this. That is, there is no larger set in $\mathbb{Z}$ containing $\mathbb{A}$ ($\mathbb{B}$) with all pairwise differences having ”odd” (”even”) exponents.

This extends to the limits

$|\alpha - \alpha'|_2 = 2^{-2k-1}$ for some k, $\alpha \neq \alpha' \in \bar{\mathbb{A}}$.

$|\beta - \beta'|_2 = 2^{-2k}$ for some k, $\beta \neq \beta' \in \bar{\mathbb{B}}$.

The closures are again maximal but this time in $\bar{\mathbb{Z}}_2$.

Note, the abuse of language ”even (odd) differences”.

Examples of Types of Results

Fix $\beta \in \bar{\mathbb{B}}$. (omit the orbit of 0.)

Define a new sequence $\mathbb{B}_\beta = SFS\{(-1)^{\beta_{2k}} \cdot 2^{2k}\}$.

Let $T_{\bar{\mathbb{B}}}$ denote the induced transformation on $\bar{\mathbb{B}}$.

Then the orbit of β under $T_{\bar{\mathbb{B}}}$ is $\{\beta + b : b \in \mathbb{B}_\beta\}$

We can also construct the skyscraper above $\bar{\mathbb{B}}_\beta$ under the map $T = +1$. Again we have a ”natural” projection of the Haar measure which extends onto the skyscraper.

This is isomorphic to the skyscraper over $\bar{\mathbb{B}}$. The isomorphism is given by subtracting β.

We also have $\mathbb{A} \oplus \mathbb{B}_\beta = \mathbb{Z}$.

Returning to the exhaustive weakly wandering sequence $\mathbb{A}$ and the exhaustive weakly wandering set $\bar{\mathbb{B}}$, we see they are both in the $\mathbb{Z}_2$.

We can reverse their roles.

Take the closure $\bar{\mathbb{A}}$. Project the Haar measure onto it. Build a skyscraper above it, and we again have an infinite measure preserving transformation $(X_{\mathbb{A}}, T)$. This time the exhaustive weakly wandering sequence is $\mathbb{B}$.

The two maps are ”dual”. Specifically, they are dissimilar meaning that there is no conservative joining of them.

By the way, the only common points in the two spaces is the orbit of 0 which is the integers.

Let's examine $3 \cdot \mathbb{B}$.

Recall $\mathbb{B}$ has "even differences" and is maximal in $\mathbb{Z}$ with repect to this property. It's closure also has "even differences" and is maximal in $\mathbb{Z}_2$ with this property.

The number 3 is a Unit in $\mathbb{Z}_2$. So when we multiply by it, it does not change the "even differences" - and - the closure $\overline{3 \cdot \mathbb{B}}$ is still maximal with repect to this property.

However, it may happen (and it does in this case) that $3 \cdot \mathbb{B}$ is no longer maximal in $\mathbb{Z}$.

This is "simply" explained by the fact that $\bar{\mathbb{B}}$ contains $-\frac{1}{3}$.

$\overline{3 \cdot \mathbb{B}}$ also has a "natural" measure and we can again build the skyscraper above it. This map is **not** isomorphic to the Hajian-Kakutani example.

$$Y = \bigcup_{k=-\infty}^{\infty} T^k \overline{3 \cdot \mathbb{B}} = \bigcup_{k=-\infty}^{\infty} 3 \cdot \bar{\mathbb{B}} + k$$

Multiply by $\frac{1}{3}$

$$\frac{1}{3} Y = \bigcup_{k=-\infty}^{\infty} \bar{\mathbb{B}} + \frac{k}{3}$$

Which is then easy to picture by tripling each level in the original Hajian-Kakutani example.