A Direct Sum Decompostion of the Integers and a Question of Y. Ito

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A counter example to a conjecture of Y. Ito concerning direct summands of the integers is presented.

1 Introduction

This paper continues the studies begun in [10], [6] and [5] of direct sum decompositions of the integers $\mathbb{Z} = \mathbb{A} \oplus \mathbb{C}$ where

$$\mathbb{A} = \left\{ \sum_{i \geq 0} \epsilon_i 2^{2i+1} \ : \ \epsilon_i \in \{0,1\} \text{ and } \epsilon_i = 1 \ \textit{for finitely many i's} \right\}$$

and the sum is understood to be unique, *i.e.* $a+c = a'+c' \Rightarrow a = a'$ and $c = c'$.

The general problem of characterizing complementing pairs of $\mathbb{Z}$ arose in the work of de Bruijn in 1950. That there is no effective characterization of all pairs for $\mathbb{Z}$ was shown by Swenson [9] (see also Post [8]). This contrasts with $\mathbb{N}$ for which a nice characterization exists. Two infinite subsets $\mathbb{C}$ and $\mathbb{D}$ are a complementing pair for the nonnegative integers $\mathbb{N}$ if and only if there exists a sequence of integers $m_0 = 1$ and $m_i \geq 2$ for all $i \geq 1$ such that $\mathbb{C}$ and $\mathbb{D}$ are the sets of all finite sums respectively $c = \sum x_{2i} M_{2i}$ and $d = \sum x_{2i+1} M_{2i+1}$ where $M_i = \prod_{j=0}^{i} m_j$ and $0 \leq x_i < m_{i+1}$. (see [10] for further references). Note, that 0 is in both corresponding to the empty sum, and $1 \in \mathbb{C}$.

The set $\mathbb{A}$ above, is one of the simpliest direct summands of $\mathbb{N}$, arising when $m_i \equiv 2$ for all $i \geq 1$. Many of the results in this paper may be extended though the definitions need to be appropriately modified. The papers [10], [6] and [5] have all worked toward characterizing the complements of $\mathbb{A}$ in $\mathbb{Z}$. We refer the reader to [10] and [2] and references therein for related work and questions in the case one of the summands is finite.

2 Previous results

The set $\mathbb{A}$ is fixed throughout the paper as defined in the previous section. Denote by $\mathfrak{C}(\mathbb{A})$ the family of all complements of $\mathbb{A}$.

The following two conditions are necessary for a set $\mathbb{C} \in \mathfrak{C}(\mathbb{A})$ (see [10] and [3]).

Conditions 2.1

(i) For every $c, c' \in \mathbb{C}$ either $c = c'$ or the maximal number i such that 2^i divides $c - c'$ is even,

(ii) C is maximal with respect to (i). That is if $\mathbb{C}'$ satisfies (i) and $\mathbb{C} \subset \mathbb{C}'$ then $\mathbb{C} = \mathbb{C}'$,

Clearly $\mathbb{C}$ is a complement if and only if $1 + \mathbb{C} = \{1 + c \ : \ c \in \mathbb{C}\}$ is a complement. So we make the normalizing simplification that $0 \in \mathbb{C}$. This implies for each $c \in \mathbb{C}$ the maximal i such that 2^i divides c is even.

One obvious complement [7] for $\mathbb{A}$ in $\mathbb{Z}$ is $-\mathbb{B}$ where

$$\mathbb{B} = \left\{ \sum_{i \geq 0} \epsilon_i 2^{2i} \ : \ \epsilon_i \in \{0, 1\} \text{ and } \epsilon_i = 1 \ for\ finitely\ many\ i's \right\}$$

In [10], this complement was used to obtain the following.

Theorem 2.2 (Tijdeman) *Let $\mathbb{C}$ be a subset of $\mathbb{Z}$ containing 0. Then $\mathbb{C} \in \mathfrak{C}(\mathbb{A})$ if and only if $\mathbb{C}$ satisfies the three conditions (i), (ii) and*
(iii) $\mathbb{A} \oplus \mathbb{C} \supset -\mathbb{B}$.

A family of complements of $\mathbb{A}$ are

$$\mathbb{B}_\omega = \left\{ \sum_{i \geq 0} \epsilon_i \omega_i 2^{2i} \ : \ \epsilon_i \in \{0, 1\} \text{ and } \epsilon_i = 1 \ for\ finitely\ many\ i's \right\}$$

where $\omega \in \{-1, 1\}^{\mathbb{N}}$ and $\omega_i = -1$ for infinitely many $i's$.

These complements were used in [6] to obtain the following.

Theorem 2.3 (Ito) *Let* $\mathbb{C}$ *be a subset of* $\mathbb{Z}$ *containing* 0. *Then* $\mathbb{C} \in \mathfrak{C}(\mathbb{A})$ *if and only if* $\mathbb{C}$ *satisfies the three conditions (i), (ii) and*

(iv) There exists an ω *as above such that* $\mathbb{A} \oplus \mathbb{C} \supset \mathbb{B}_\omega$

In [5], Dateyama and Kamae extended the family $\{\mathbb{B}_\omega\}$ of complements and similarly extended the result.

Let $\psi = \{\psi_n\}_{n\geq 0}$ be a set of maps $\psi_n : \{-1,0,1\}^{\mathbb{N}} \to \{-1,1\}$ such that for any $(\epsilon_0, \epsilon_1, \cdots) \in \{-1,0,1\}^{\mathbb{N}}$, $\psi_n(\epsilon_0, \epsilon_1, \cdots, \epsilon_{n-1}) = -1$ for infinitely many $n's$. ψ_0 is a constant of value ± 1. Define

$$\mathbb{B}_\psi = \left\{ \sum_{i\geq 0} \epsilon_i 2^{2i} \ : \ \text{finite sums where} \ \ \epsilon_i = 0 \text{ or } \epsilon_i = \psi_i(\epsilon_0, \cdots, \epsilon_{i-1}) \right\}$$

Theorem 2.4 (Dateyama and Kamae) *Let* $\mathbb{C}$ *be a subset of* $\mathbb{Z}$ *containing* 0. *Then* $\mathbb{C} \in \mathfrak{C}(\mathbb{A})$ *if and only if* $\mathbb{C}$ *satisfies the three conditions (i), (ii) and*

(v) There exists a ψ *as above such that* $\mathbb{A} \oplus \mathbb{C} \supset \mathbb{B}_\psi$

Conjecture 2.5 *In [6], it was conjectured that the third condition in the above theorems could be replaced with*

(vi) There exists a $\mathbb{D} \in \mathfrak{C}(\mathbb{A})$ *such that* $\mathbb{A} \oplus \mathbb{C} \supset \mathbb{D}$.

We present a counter example to this in the section 4.

3 2-adics

In this section, we present and discuss some results on the 2-adic integers which will be used in the sequel.

Let

$$\mathbb{Z}_2 = \left\{ z = \sum_{i\geq 0} z_i 2^i \ : \ z_i \in \{0,1\} \right\}$$

denote the completion of $\mathbb{Z}$ in the 2-adic valuation norm. For notational convenience we identify $\mathbb{Z}_2$ with $\{0,1\}^{\mathbb{N}}$, *i.e.* $z = \sum z_i 2^i \leftrightarrow (z_0, z_1, z_2, \cdots)$. The positive integers are represented by $n = (z_0, z_1, z_2, \cdots)$ with $z_i = 0$ for all but finitely many i's. The negative integers are represented by $m = (z_0, z_1, z_2, \cdots)$ with $z_i = 1$ for all but finitely many i's.

As usual $\text{ord}(n) = \text{ord}_2(n)$ is the highest power of 2 which divides n. This extends to all $z = (z_0, z_1, z_2, \cdots) \in \mathbb{Z}_2$ by $\text{ord}_2(z) = i$ where $z_i = 1$ and $z_j = 0$

for all $0 \le j < i$. The ord is used in analyzing the distance of two numbers, that is $\text{ord}(c-d) = n$ means that c, d are the same for the first n coordinates $c_i = d_i$ for $0 \le i \le n-1$. We will often be concerned with whether the ord is even or odd. Note that $\text{ord}_2(0) = \infty$ and this is considered both odd and even.

Recalling conditions 2.1, a subset $\mathbb{E}$ of $\mathbb{Z}_2$ is said to have **even differences** if $\text{ord}_2(e - e')$ is even for all $e \neq e'$. (**Odd differences** is defined similarly.). A set of integers $\mathbb{C}$ which has even differences is said to be **maximal in** $\mathbb{Z}$ if it satisfies (ii) of 2.1. A subset $\mathbb{E}$ of $\mathbb{Z}_2$ with even differences is **maximal in** $\mathbb{Z}_2$ if any subset containing $\mathbb{E}$ with even differences coincides with $\mathbb{E}$. We will use the term "maximal" when it is clear from the context which definition applies. (Similar definitions hold for odd differences.)

A set of integers $\mathbb{C}$ is **even complete** if for all $n \ge 1$ and for every $\xi = (\xi_0, \xi_1, \cdots, \xi_{n-1}) \in \{0,1\}^{\mathbb{N}}$ there exists a $c \in \mathbb{C}$ with $c_{2i} = \xi_i$, $0 \le i \le n-1$. Similarly a set of integers $\mathbb{D}$ is **odd complete** if if for all $n \ge 1$ and for every $\xi = (\xi_0, \xi_1, \cdots, \xi_{n-1}) \in \{0,1\}^{\mathbb{N}}$ there exists a $d \in \mathbb{D}$ with $d_{2i+1} = \xi_i$, $0 \le i \le n-1$.

Lemma 3.1 *Let $\mathbb{C}$ be a set of integers containing 0 which have even differences and is maximal in $\mathbb{Z}$. Then $\mathbb{C}$ is even complete.*

Proof. This is essentially contained in Lemma 1 in [5] which proves a bit more. Let $n \ge 1$ be the smallest integer such that there exists $(\xi_0, \cdots, \xi_{n-1})$ and no $c \in \mathbb{C}$ with $c_{2i} = \xi_i$, $0 \le i \le n-1$. If $n = 1$ there are two cases depending on the value of ξ_0. If $\xi_0 = 1$ then $\mathbb{C}$ contains no odd integers and 1 may be adjoined to $\mathbb{C}$ and maintain even differences. If $\xi_0 = 0$ it means all integers in $\mathbb{C}$ are odd and the number 4 may be adjoined. If $n > 1$ let $c \in \mathbb{C}$ with $c_{2i} = \xi_i$, $0 \le i \le n-2$. Hence $c_{2n-2} \neq \xi_{n-1}$ and $\text{ord}(c - c') \neq 2n-2$ for all $c' \in \mathbb{C}$. The number $c + 2^{2n-2}$ may then be adjoined to $\mathbb{C}$ as $\text{ord}(c + 2^{2n-2} - c') = \min(\text{ord}(c - c'), 2n-2)$.

The two conditions even differences and even complete are not enough to make a set a complement of $\mathbb{A}$ or even maximal. Consider the set $-\mathbb{B}$ and remove from it the number -1. This is still even complete but is not maximal with respect to even differences and it is not a complement. Observe however that -1 is in the closure of this set.

The following two lemmas appear in [4] in a more general form and are variations of Lemma 3 in [5].

Lemma 3.2 *Let $\mathbb{C}$ be a set of integers containing 0 which has even differences and is even complete. Then $\bar{\mathbb{C}}$ has even differences and is maximal with respect to even differences in $\mathbb{Z}_2$. That is if $\mathbb{C}' \supset \bar{\mathbb{C}}$ and $\mathbb{C}'$ has even differences then $\mathbb{C}' = \bar{\mathbb{C}}$.*

The corresponding result for odd differences in place of even differences also holds.

Proof. Let $z, z' \in \bar{\mathbb{C}}$ with $\mathrm{ord}(z - z') = n$. Choose $c, c' \in \mathbb{C}$ with $\mathrm{ord}(c - z) > n$ and $\mathrm{ord}(c' - z') > n$. Hence $\mathrm{ord}(c - c') = n$ and so is even.

Suppose $\mathrm{ord}(z - x)$ is even for all $z \in \bar{\mathbb{C}}$. Put $\xi_i = x_{2i}$, $i \geq 0$. Then for each $n \geq 0$, by the definition of even complete, there must be a $c_n \in \mathbb{C}$ with $(c_n)_{2i} = x_{2i}$ for $0 \leq i < n$. By even differences of $c_n - x$, $(c_n)_j = x_j$, $0 \leq j \leq 2n$. Therefore c_n converge to x and x is in the closure of $\mathbb{C}$.

Lemma 3.3 *Let $\mathbb{C}$ be a set of integers containing 0 which has even differences and is even complete. Let $\mathbb{E}$ be a set of integers containing 0 which has odd differences and is odd complete. Then*

$$\overline{\mathbb{C} \oplus \mathbb{E}} = \bar{\mathbb{C}} \oplus \bar{\mathbb{E}} = \mathbb{Z}_2$$

Proof. Even and odd differences make the sums unique. If $c + d = c' + d'$ then $c - c' = d - d'$. Hence this difference is both even and odd and so must be 0. The denseness of $\mathbb{C} \oplus \mathbb{E}$ is similar to the reasoning in the previous proof. The equality of the closure of the sum with the sum of the closures is straightforward from the odd/even differences.

Lemma 3.2 supplies a converse to Lemma 3.1.

Lemma 3.4 *Let $\mathbb{C}$ be a set of integers containing 0 which has even differences and is even complete. Then $\mathbb{C}' = \bar{\mathbb{C}} \cap \mathbb{Z}$ has even differences and is maximal in $\mathbb{Z}$.*

Remark 3.5 Lemma 3.3 clarifies how a set $\mathbb{C}$ can satisfy conditions 2.1 yet not be a complement of $\mathbb{A}$ in $\mathbb{Z}$. For any integer n which is not in $\mathbb{A} \oplus \mathbb{C}$ there must be an $\bar{a} \in \bar{\mathbb{A}} \backslash \mathbb{A}$ and a $\bar{c} \in \bar{\mathbb{C}} \backslash \mathbb{C}$ so that $\bar{a} + \bar{c} = n$. Observe that any $\bar{a} = (a_0, a_1, \cdots) \in \bar{\mathbb{A}}$ has $1's$ only in odd locations, *i.e.* $a_{2i} = 0$ for all i, and $\bar{a} \in \bar{\mathbb{A}} \backslash \mathbb{A}$ means $a_{2i+1} = 1$ for infinitely many i. As an illustration consider the set $\mathbb{B}$ which satisfies conditions 2.1 but is not a complement.

The numbers $-1/3 = (1,0,1,0,\overline{1,0}) \in \bar{\mathbb{B}}$ and $-2/3 = (0,1,0,1,\overline{0,1}) \in \bar{\mathbb{A}}$ and so -1 is not be in $\mathbb{A} \oplus \mathbb{B}$. (That both $\mathbb{A}$ and $\mathbb{B}$ are positive and so obviously the sum contains no negative integers is a red herring in understanding the situation.

Lemma 3.6 *Let $\mathbb{C}$ be a subset of $\mathbb{Z}$ containing 0. Then $\mathbb{C} \in \mathfrak{C}(\mathbb{A})$ if and only if $\mathbb{C}$ satisfies the three conditions (i), (ii) and*
(v) For any $\bar{c} = (c_0, c_1, \cdots) \in \bar{\mathbb{C}} \backslash \mathbb{C}$ $c_{2i} = 0$ for infinitely many i.

Proof. Assume $\mathbb{C}$ satsifies conditions (i) and (ii). By maximality and Lemma 3.2 $\bar{c} \in \bar{\mathbb{C}} \backslash \mathbb{C}$ is not an integer. Therefore there are infinitely many i with $c_i = 0$ and infinitely many with $c_i = 1$.

Suppose it has only finitely many i such that $c_{2i} = 0$. Then there exist an $n > 0$ such that if $i \geq n$ and $c_i = 0$ then i must be odd. Denote the collection of these i as I. Define $\bar{a} \in \bar{\mathbb{A}} \backslash \mathbb{A}$ by $a_i = 1$ for all $i \in I$ and no where else. Then $\bar{a} + \bar{c}$ is a negative integer and $\mathbb{C}$ cannot be a complement.

Suppose there are infinitely many i with $c_{2i} = 0$. Denote this set of i as I. We claim that there is no $\bar{a} \in \bar{\mathbb{A}}$ with $\bar{a} + \bar{c}$ an integer. In order for $\bar{a} + \bar{c}$ to be a negative integer it must have a 1 in all but a finite number of the coordinates $i \in I$. Since these are even there must have been a carry from a lower coordinate. Consider $i < j$, $i, j \in I$ such that there is no $k \in I$ with $i < k < j$. There can be no carry from the $2i^{th}$ coordinate. Hence to get a carry into the $2j^{th}$ coordinate there must be an odd coordinate $2i < 2k + 1 < 2j$ which starts the carry. But then the $2k + 1$ coordinate of $\bar{a} + \bar{c}$ must be 0 and the sum cannot be an integer. A similar argument shows that the sum cannot be a postive integer.

4 Example

In this section we will construct two subsets of the integers $\mathbb{C}$ and $\mathbb{D}$ which both satisfy conditions 2.1. The set $\mathbb{C}$ will not be a complement, the set $\mathbb{D}$ will be a complement and $\mathbb{A} \oplus \mathbb{C} \supset \mathbb{D}$. These sets are a variation of Example 4.2 appearing in [6] and are both built from the same general construction.

Define

$$\mathbb{C}_{\mathbf{p}} = \cup_{i \geq 1} \{p_i - 2^{2i}\mathbb{B}\} = \cup_{i \geq 1} \{p_i - 2^{2i} b \mid b \in \mathbb{B}\}$$

where the set $\mathbb{B}$ is as defined in Section 2, and $\mathbf{p} = \{p_k\}$, $k \geq 1$, is a sequence of integers satisfying

(i) $p_1 = 0$,
(ii) $1 \le p_i < 2^{2i}$ is an odd integer for all $i \ge 2$,
(iii) $\mathrm{ord}_2(p_i - p_{i+1}) = 2(i-1)$.

The utility of this construction is evidenced by the following.

Lemma 4.1

1. *p_i converge to some $\bar{p}$ in $\mathbb{Z}_2$,*

2. *$\mathbb{C}_{\mathbf{p}}$ has even differences,*

3. *$\mathbb{C}_{\mathbf{p}}$ is even complete,*

4. *$\mathbb{C}_{\mathbf{p}}$ is maximal in $\mathbb{Z}$ if and only if $\bar{p}$ is not an integer,*

5. *$\mathbb{C}_{\mathbf{p}}$ is a complement if and only if $\bar{p}$ is not an integer and there does not exist an $\bar{a} \in \mathbb{A}$ with $\bar{a} + \bar{p}$ an integer.*

We first present a few examples before proving the lemma including the two sets for the counter example. (Because of 1, redenote $\mathbb{C}_{\mathbf{p}}$ as $\mathbb{C}_{\bar{p}}$.)

Example 1 *$p_k = \sum_{i=0}^{k-2} 2^{2i}$ for $k \ge 2$.*

This appears in [6]. A few of the representations of these are $p_1 = (0, \bar{0})$, $p_2 = (1, 0, \bar{0})$, $p_3 = (1, 0, 1, 0, \bar{0})$, $p_4 = (1, 0, 1, 0, 1, 0, \bar{0})$, and it is easy to see that the limit is $\bar{p} = (1, 0, \overline{1, 0}) = -1/3$. Since $-2/3 = (0, 1, \overline{0, 1}) \in \bar{\mathbb{A}}$, the set $\mathbb{C}_{-1/3}$ is not a complement.

Example 2 *$p_k = \sum_{i=0}^{k-2} (2^{2i} + 2^{2i+1})$ for $k \ge 2$.*

A few of the representations of these are $p_1 = 0$, $p_2 = (1, 1, 0, \bar{0})$, $p_3 = (1, 1, 1, 1, 0, \bar{0})$, $p_4 = (1, 1, 1, 1, 1, 1, 0, \bar{0})$ and the limit is $\bar{p} = (1, \bar{1}) = -1$. In this case, $\bar{\mathbb{C}}_{-1}$ is not maximal as well as not a complement.

Example 3 *$p_k = \sum_{i=0}^{k-2} 3^{(i+1) \bmod 2} \cdot 2^{2i}$ for $k \ge 2$.*

A few of the representations of these are $p_1 = 0$, $p_2 = (1, 1, 0, \bar{0})$, $p_3 = (1, 1, 1, 0, \bar{0})$, $p_4 = (1, 1, 1, 0, 1, 1, 0, \bar{0})$, $p_5 = (1, 1, 1, 0, 1, 1, 1, 0, \bar{0})$ and the limit is $\bar{p} = (1, 1, 1, 0, \overline{1, 1, 1, 0}) = -7/15$. $\mathbb{C}_{-7/15}$ is not a complement because $-8/15 = (0, 0, 0, 1, \overline{0, 0, 0, 1}) \in \bar{\mathbb{A}}$.

Example 4 $p_k = \sum_{i=0}^{k-2} \left(3^{(i+1) \bmod 2} \cdot 2^{2i} + ((i+1) \bmod 2) \cdot 2^{2i+1}\right)$ *for* $k \geq 2$.

A few of the representations of these are $p_1 = 0$, $p_2 = (1,0,1,0,\bar{0})$, $p_3 = (1,0,0,1,0,\bar{0})$, $p_4 = (1,0,0,1,1,0,1,0,\bar{0})$, $p_5 = (1,0,0,1,1,0,0,1,0,\bar{0})$ and the limit is $\bar{p} = (1,0,0,1,\overline{1,0,0,1}) = -9/15$. From the above Lemma as well as Lemma 3.6 it follows that $\mathbb{C}_{-9/15}$ is a complement.

Counter Example. The sets $\mathbb{C}_{-7/15}$ and $\mathbb{C}_{-9/15}$ form the promised counter example. To see that $\mathbb{A} \oplus \mathbb{C}_{-7/15} \supset \mathbb{C}_{-9/15}$ simply observe that the difference of the p_k for $\mathbb{C}_{-9/15}$ and $\mathbb{C}_{-7/15}$ is $q_k = \sum_{i=o}^{k-2}((i+1) \bmod 2) \cdot 2^{2i+1} \in \mathbb{A}$.

Proof of Lemma 4.1.

1 and 2 are clear from the definition.

To see 3 begin by observing that $-\mathbb{B}$ is even complete. Hence for all $(\xi_0, \cdots, \xi_{n-1})$ with $\xi_1 = 0 = (p_1)_0$ there is a $c \in \{p_1 - 2^2 \cdot \mathbb{B}\}$ with $c_{2i} = \xi_i$, $0 < i \leq n-1$.

Next look at all $(\xi_0, \cdots, \xi_{n-1})$ with $\xi_0 = 1 = (p_2)_0$ and $\xi_2 = (p_2)_2$. Since $1 \leq p_2 < 2^4$ it is clear that for each of these patterns there is a $c \in \{p_2 - 2^4 \cdot \mathbb{B}\}$ with $c_{2i} = \xi_i$, $0 < i \leq n-1$.

We don't know what $(p_2)_2$ is (either 0 or 1), but we have by assumption $\mathrm{ord}(p_2 - p_3) = 2^{2\cdot(2-1)} = 2^2$. This means that $(p_3)_0 = (p_2)_0$ and $(p_3)_2 = ((p_2)_2 + 1) \bmod 2$. Hence for each $(\xi_0, \cdots, \xi_{n-1})$ with $\xi_1 = (p_3)_0$, $\xi_2 = (p_3)_2$ and $\xi_3 = (p_3)_4$ there is a $c \in \{p_3 - 2^6 \cdot \mathbb{B}\}$ with $c_{2i} = \xi_i$, $0 < i \leq n-1$.

It is easy to see that the proof of even completeness continues by induction.

4 follows by Lemmas 3.2 and 3.4. First observe that if $\bar{c} \in \bar{\mathbb{C}}_{\bar{p}} \backslash \mathbb{C}_{\bar{p}}$ then either $\bar{c} \in \{p_i - 2^{2i}\mathbb{B}\}$ for some i or $\bar{c} = \bar{p}$. It is clear that $\overline{p_i - 2^{2i}\mathbb{B}}$ contains no integers so the only possible additional integer in $\bar{\mathbb{C}}_{\bar{p}}$ can be $\bar{p}$.

Finally 5 follows by Remark 5. This completes the proof.

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