

Hecke Algebras and Representations

Sachin Gautam

1 Generic Algebras

Let (W, S) be a Coxeter System. Let k be a commutative ring with identity. Let \mathcal{E} denote free k -module over $\{T_w\}_{w \in W}$. Fix a family of elements of k , $(a_s, b_s)_{s \in S}$, such that, whenever s and s' are conjugate to each other we have

$$a_s = a_{s'} \quad b_s = b_{s'}$$

For such a setting, we want to define a structure of k -algebra over \mathcal{E} , such that following conditions hold for every $s \in S$ and $w \in W$:

$$T_s \cdot T_w = \begin{cases} T_{sw} & \text{if } sw > w \\ a_s T_w + b_s T_{sw} & \text{if } sw < w \end{cases} \quad (1)$$

First main theorem is that such an algebra structure exists on \mathcal{E} .

Theorem 1 *For a given family of elements of k , $(a_s, b_s)_{s \in S}$, there exists unique k -algebra structure on \mathcal{E} , such that relations of (1) are satisfied. We denote this k -algebra by $\mathcal{E}_k((a_s), (b_s))$.*

Remark 1 *Relations (1) can also be formulated in right handed fashion as follows: For every $t \in S$ and $w \in W$, we have,*

$$T_w \cdot T_t = \begin{cases} T_{wt} & \text{if } wt > w \\ a_t T_t + b_t T_{wt} & \text{if } wt < w \end{cases} \quad (2)$$

To prove this claim, let $w \in W$ and $t \in S$. If $wt > w$, choose $s \in S$, such that $sw < w$. This implies $l(w) = l(swt) > l(sw)$. By induction on $l(w)$, we can get

$$T_{sw} T_t = T_{swt}$$

which when multiplied by T_s on left, gives

$$T_s T_{sw} T_t = T_s T_{swt} \Rightarrow T_w T_t = T_{wt}$$

On the other hand, if we have $wt < w$, then above relation can be applied to T_{wt} to get

$$T_{wt} T_t = T_w$$

Multiply this equation on both sides, by T_t on right.

$$T_w T_t = T_{wt} T_t^2 = T_{wt}(a_t T_t + b_t T_1) = a_t T_w + b_t T_{wt}$$

and hence the relations (2).

Remark 2 Yet another pair of equivalent relations are:

$$T_s^2 = a_s T_s + b_s T_1 \tag{3}$$

$$T_s T_w = T_{sw} \text{ if } sw > w \tag{4}$$

These relations are evidently consequence of (1). To prove other side of implication, we only need to consider the case when $sw < w$. In this case, $s(sw) > sw$. Therefore (4) implies that $T_s T_{sw} = T_w$. Therefore,

$$T_s T_w = T_s^2 T_{sw} = (a_s T_s + b_s T_1) T_{sw} = a_s T_w + b_s T_{sw}$$

If we assume that there exists k -algebra structure on \mathcal{E} , satisfying (1), it is easy to see that such structure is unique. These relations imply, by induction on length of w , that if $w = s_1 \dots s_p$ is reduced decomposition, then

$$T_w = T_{s_1} \dots T_{s_p}$$

Therefore, algebra structure on \mathcal{E} is completely determined by relations (1), hence uniqueness.

2 Proof of Theorem 1 (Existence)

For every $s \in S$, define the following endmorphisms of k -module \mathcal{E} :

$$\lambda_s(T_w) = \begin{cases} T_{sw} & \text{if } sw > w \\ a_s T_w + b_s T_{sw} & \text{if } sw < w \end{cases} \tag{5}$$

$$\mu_s(T_w) = \begin{cases} T_{ws} & \text{if } ws > w \\ a_s T_w + b_s T_{ws} & \text{if } ws < w \end{cases} \tag{6}$$

We will first show that these two operators commute. For this we will require the following lemma:

Lemma 1 Let $w \in W$ and $s, t \in S$. If $l(swt) = l(w)$ and $l(sw) = l(wt)$, then we have $sw = wt$.

PROOF:

We will use Partition Axioms of Coxeter Systems. Assume first that $w < sw$. In this case, we have

$$l(s(wt)) = l(w) < l(sw) = l(wt)$$

Therefore, $w \in P_s$ and $wt \notin P_s$. By (P_3) , this implies that $sw = wt$. On the other hand, if $sw < w$, then we get

$$l(s(wt)) = l(w) > l(wt) = l(sw)$$

Therefore, $wt \in P_s$ and $(wt)t \notin P_s$. Again using (P_3) , we get that $(wt)t = s(wt)$.

QED

Proposition 1 For $s, t \in S$, we have $\lambda_s \mu_t = \mu_t \lambda_s$.

PROOF:

Let $w \in W$. We explicitly compute $\lambda_s(\mu_t(T_w))$ and $\mu_t(\lambda_s(T_w))$ to see that they are equal. There are six possible cases of ordering on w, sw, wt, swt . Examining each case as follows:

(a) $l(w) < l(wt) = l(sw) < l(swt)$

In this case, $\lambda_s(\mu_t(T_w)) = T_{swt} = \mu_t(\lambda_s(T_w))$.

(b) $l(swt) < l(wt) = l(sw) < l(w)$

In this case, we have

$$\lambda_s(\mu_t(T_w)) = \lambda_s(a_t T_w + b_t T_{wt}) = a_t(a_s T_w + b_s T_{sw}) + b_t(a_s T_{wt} + b_s T_{swt})$$

$$\mu_t(\lambda_s(T_w)) = \mu_t(a_s T_w + b_s T_{sw}) = a_s(a_t T_w + b_t T_{wt}) + b_s(a_t T_{sw} + b_t T_{swt})$$

and the two resulting terms are equal since k is a commutative ring.

(c) $l(wt) = l(sw) < l(swt) = l(w)$

In this case, from previous lemma, we get that $sw = wt$ and hence $a_s = a_t$ and $b_s = b_t$.

Therefore,

$$\lambda_s(\mu_t(T_w)) = a_s^2 T_w + b_s a_s T_s w + b_s T_{swt} = \mu_t(\lambda_s(T_w))$$

(d) $l(wt) < l(w) = l(swt) < l(sw)$

In this case, we have the following:

$$\lambda_s(\mu_t(T_w)) = a_t T_{sw} + b_t T_{swt} = \mu_t(\lambda_s(T_w))$$

(e) $l(sw) < l(w) = l(swt) < l(wt)$

$$\lambda_s(\mu_t(T_w)) = a_s T_{wt} + b_s T_{swt} = \mu_t(\lambda_s(T_w))$$

(f) $l(w) = l(swt) < l(sw) = l(wt)$

This case is similar to (c), since previous lemma is again applicable.

QED

We now return to prove the existence part of Theorem 1. Let \mathcal{L} be k -subalgebra of $End_k(\mathcal{E})$, generated by $\{\lambda_s\}_{s \in S}$. We define k -module homomorphism $\varphi : \mathcal{L} \longrightarrow \mathcal{E}$ by

$$\varphi(\lambda) = \lambda(T_1)$$

φ is clearly k -module map. Moreover, for any $w \in W$, $T_w = \varphi(\lambda_{s_1} \dots \lambda_{s_p})$, for any reduced decomposition $w = s_1 \dots s_p$. This proves that φ is surjective map. To prove injectivity, assume that $\varphi(\lambda) = 0$. We will prove by induction on $l(w)$ that $\lambda(T_w) = 0$ and hence $\lambda = 0$.

For any $w \in W$, take $t \in S$, such that $wt < w$. Then $\lambda(T_w) = \lambda(T_{wt}T_t) = \lambda(\mu_t(T_{wt}))$. Since μ_t commutes with all the generators of \mathcal{L} , it also commutes with $\lambda \in \mathcal{L}$. Therefore, previous term is equal to $\mu_t(\lambda(T_{wt})) = \mu_t(0) = 0$, since by induction hypothesis, $\lambda(T_v) = 0$ for every $v \in W$, such that $l(v) < l(w)$. This proves that φ is k -module isomorphism. Therefore, we can transport structure of k -algebra on \mathcal{L} to \mathcal{E} . Explicitly

$$T_s.T_w = \varphi(\lambda_s.\lambda_w)$$

As we can talk of $\lambda_w = \lambda_{s_1} \dots \lambda_{s_p}$, which is independent of choice of reduced decomposition $w = s_1 \dots s_p$. It remains to check that this structure of k -algebra on \mathcal{E} satisfies the conditions of (1) or equivalently the relations (3) and (4). It is enough to verify these conditions on $\lambda \in \mathcal{L}$.

If $l(sw) > l(w)$, take $w = s_1 \dots s_r$ as reduced decomposition. Then $ss_1 \dots s_r$ is reduced decomposition of sw . Thus

$$\lambda_{sw} = \lambda_s \lambda_{s_1} \dots \lambda_{s_r} = \lambda_s \lambda_w$$

which is (4). To check (3), λ_s^2 can be evaluated for any T_w .

In the first case, let $w < sw$. Then $\lambda_s(\lambda_s(T_w)) = \lambda_s(T_{sw}) = a_s T_{sw} + b_s T_w$, which in turn is equal to $(a_s \lambda_s + b_s \lambda_1)(T_w)$.

Again let $sw < w$. In this case $\lambda_s(\lambda_s(T_w)) = \lambda_s(a_s T_w + b_s T_{sw}) = a_s \lambda_s(T_w) + b_s T_w$. This finishes the proof of Theorem 1.

3 Hecke Algebra and R-polynomials

Take $\mathbb{Z}[q, q^{-1}]$, ring of Laurent polynomials over \mathbb{Z} for k and $a_s = q - 1$, $b_s = q$, for every $s \in S$. The resulting algebra, denoted by \mathcal{H} is called Hecke algebra associated with Coxeter System (W, S) .

$$\mathcal{H} = \mathcal{E}_{\mathbb{Z}[q, q^{-1}]}(q - 1, q)$$

Multiplication in \mathcal{H} becomes:

$$T_s.T_w = T_{sw} \text{ if } sw > w \tag{7}$$

$$T_s^2 = (q - 1)T_s + qT_1 \tag{8}$$

This implies that $T_s(T_s - (q-1)T) = qT_1$. Therefore, T_s is invertible in \mathcal{H} , for every $s \in S$, with inverse given as:

$$T_s^{-1} = a^{-1}(T_s - (q-1)T_1)$$

As $T_w = T_{s_1} \dots T_{s_p}$, if $w = s_1 \dots s_p$ is reduced decomposition, we get $T_w^{-1} = T_{s_p}^{-1} \dots T_{s_1}^{-1}$ and therefore all T_w are invertible in \mathcal{H} .

Now fix a reduced decomposition $w = s_1 \dots s_p$. Above considerations apply to result:

$$\begin{aligned} (T_{s^{-1}})^{-1} &= T_{s_1}^{-1} \dots T_{s_p}^{-1} \\ &= q^{-l(w)} (-1)^{l(w)} ((q-1)T_1 - T_{s_1}) \dots ((q-1)T_1 - T_{s_p}) \end{aligned}$$

which gives us the second main theorem defining R -polynomials.

Theorem 2 *For any $w \in W$, we have the following (with $\varepsilon_w = (-1)^{l(w)}$ and $q_w = q^{l(w)}$):*

$$(T_{w^{-1}})^{-1} = \varepsilon_w q_w^{-1} \sum_{x \leq w} \varepsilon_x R_{x,w} T_x \quad (9)$$

where $' \leq'$ is Bruhat-Chevalley ordering and $R_{x,w} \in \mathbb{Z}[q]$, with degree equal to $l(w) - l(x)$. Moreover, $R_{w,w} = 1$, for every $w \in W$.

PROOF:

We prove the claim (9) by induction on $l(w)$. Base case is contained in following statement already proved:

$$T_s^{-1} = -q^{-1}((q-1)T_1 - T_s)$$

which gives R -polynomials $R_{1,s} = q-1$ and $R_{s,s} = 1$.

For any given $w \in W$, assume by induction hypothesis that claim (9) holds for every $v \in W$, with $l(v) < l(w)$. Take $s \in S$ such that $sw < w$ and substitute $v = sw$.

$$\begin{aligned} (T_{w^{-1}})^{-1} &= (T_{v^{-1}} T_s)^{-1} = T_s^{-1} T_{v^{-1}}^{-1} \\ &= -q^{-1}((q-1)T_1 - T_s) \varepsilon_v q_v^{-1} \sum_{x \leq v} \varepsilon_x R_{x,v} T_x \\ &= \varepsilon_w q_w^{-1} \left(\sum_{x \leq v} (q-1) \varepsilon_x R_{x,v} T_x - \sum_{x \leq v} \varepsilon_x R_{x,v} T_s \cdot T_x \right) \end{aligned}$$

We want to get the last term as single summation. For any $x \leq w$, we have the following possibilities:

Case 1 $sx < x$. This automatically implies that $sx < sw = v$. Let $y = sx < v$, therefore, $x = sy > y$ and hence T_x appears as T_{sy} in last term whose coefficient becomes:

$$-\varepsilon_y R_{y,v} = \varepsilon_x R_{sx,sw}$$

Moreover, even if $x \leq v$, then x doesn't contribute towards coefficient of T_x in $(T_{w^{-1}})^{-1}$. As $sx < x$, the term corresponding to $x \leq v$ becomes

$$(q-1)\varepsilon_x R_{x,v} T_x - ((q-1)\varepsilon_x R_{x,v} T_x + q\varepsilon_x R_{x,v} T_{sx})$$

where T_x gets cancelled.

Therefore, we can define $R_{x,w} := R_{sx,sw}$. It is then easy to check that if $x = w$, $R_{w,w} = R_{sw,sw} = 1$, by induction hypothesis and $\deg(R_{x,w}) = \deg(R_{sx,sw}) = l(w) - 1 - l(x) + 1 = l(w) - l(x)$.

Case 2 $sx > x$. We have to consider two subcases, namely when $sx \leq v$ and when $sx \not\leq v$.

(a) $x < sx \leq v$. In this case, T_x appears in expansion as T_x as well as T_{sy} , where $y = sx$. Therefore, coefficient of T_x becomes

$$(q-1)\varepsilon_x R_{x,v} - q\varepsilon_y R_{y,v}$$

Therefore, we can define $R_{x,w} := (q-1)R_{x,sw} + qR_{sx,sw}$. We only have to check the degree of $R_{x,w}$ satisfies conditions stated in theorem. It is clear that $\deg(R_{x,w}) = \max(1+l(w) - 1 - l(x), 1+l(w) - 1 - l(x) - 1)$ which is same as $l(w) - l(x)$.

(b) When $sx \not\leq v$, we can again define $R_{x,w}$ as we did in previous subcase, if we agree to convention that $R_{x,v} = 0$ whenever $x \not\leq v$. This completes the proof of theorem.

QED

This proof also give neat procedure for computing R -polynomials. We briefly describe the steps to employ in order to find recursive relation for computing them. Assume that $x \leq w$ are given and we want to find $R_{x,w}$.

Fix $s \in S$ such that $sw < w$.

$$R_{x,w} = R_{sx,sw}, \text{ if } sx < x \tag{10}$$

$$R_{x,w} = (q-1)R_{x,sw} + qR_{sx,sw}, \text{ if } sx > x \tag{11}$$

Observe that we could have done the proof equally well by selecting $s \in S$, such that $ws < w$. We will thus write right handed versions of above steps, symmetrically. Fix $s \in S$ such that $ws < w$.

$$R_{x,w} = R_{xs,ws}, \text{ if } xs < x \tag{12}$$

$$R_{x,w} = (q-1)R_{x,ws} + qR_{xs,ws}, \text{ if } xs > x \tag{13}$$

4 An involution on \mathcal{H}

Define $\iota : \mathcal{H} \longrightarrow \mathcal{H}$ by sending q to q^{-1} and

$$\iota(T_w) = (T_{w^{-1}})^{-1}$$

and extend it linearly. We will prove that $\iota^2 = Id_{\mathcal{H}}$. First we check it on T_s for any $s \in S$.

$$\begin{aligned} \iota^2(T_s) &= \iota(T_s^{-1}) = \iota(q^{-1}(T_s - (q-1)T_1)) \\ &= q(T_s^{-1} - (q^{-1}-1)T_1) \\ &= q(q^{-1}T_s - (1-q^{-1})T_1 + (1-q^{-1})T_1) \\ &= T_s \end{aligned}$$

Thus to prove that $\iota^2 = Id_{\mathcal{H}}$, we only need to check that ι is algebra map. That is, $\iota(T_s T_w) = \iota(T_s)\iota(T_w)$. We verify this in following steps

If $sw > w$, then we have $\iota(T_s T_w) = \iota(T_{sw}) = (T_{w^{-1}s})^{-1} = T_s^{-1}(T_{w^{-1}})^{-1}$, which is same as $\iota(T_s)\iota(T_w)$.

If $sw < w$, then we substitute $v = w^{-1}s$, such that $w^{-1} = vs$ is reduced decomposition. Thus we get following steps:

$$\begin{aligned}\iota(T_s T_w) &= \iota(qT_{sw} + (q-1)T_w) \\ &= q^{-1}T_v^{-1} + (q^{-1}-1)(T_{w^{-1}})^{-1} \\ &= q^{-1}T_v^{-1} + (q^{-1}-1)(T_s^{-1}T_v^{-1}) \\ &= q^{-1}T_v^{-1} + q^{-1}(q^{-1}-1)(T_s - (q-1)T_1)T_v^{-1} \\ &= q^{-2}(q^2 - q + 1)T_v^{-1} - (q-1)q^{-2}T_s T_v^{-1}\end{aligned}$$

Also $\iota(T_s)\iota(T_w) = T_s^{-1}(T_{w^{-1}})^{-1} = T_s^{-2}T_v^{-1}$. We can thus evaluate T_s^{-2} , as follows:

$$\begin{aligned}T_s^{-2} &= q^{-2}(T_s - (q-1)T_1)^2 \\ &= q^{-2}((q-1)T_s + qT_1 - 2(q-1)T_s + (q-1)^2T_1) \\ &= q^{-2}((q^2 - q + 1)T_1 - (q-1)T_s)\end{aligned}$$

which proves the claim that $\iota(T_s T_w) = \iota(T_s)\iota(T_w)$. The rest part follows from induction, which completes the prove of assertion that ι is involution on Hecke algebra.

5 Further Properties of R -polynomials

For sake of notation we will use $\bar{R}_{x,w} := \iota(R_{x,w})$ or in other words, we can say

$$\bar{R}_{x,w}(q) = R_{x,w}(q^{-1})$$

Proposition 2 *For every $x, w \in W$, we have the following:*

- (a) $\bar{R}_{x,w} = \varepsilon_x \varepsilon_w q_x q_w^{-1} R_{x,w}$
- (b) $(T_{w^{-1}})^{-1} = \sum_{x \leq w} q_x^{-1} \bar{R}_{x,w} T_x$
- (c) $\sum_{x \leq y \leq w} \varepsilon_x \varepsilon_y R_{x,y} R_{y,w} = \delta_{x,w}$

PROOF:

(a) We prove this part by induction on $l(w)$, assuming $x \leq w$. As already seen in proof of Theorem 9, we have two cases to consider, for $s \in S$, such that $sw < w$.

If $sx < x$, we have $R_{x,w} = R_{sx,sw}$. Applying involution ι , we get

$$\begin{aligned}\bar{R}_{x,w} &= \bar{R}_{sx,sw} \\ &= \varepsilon_{sx} \varepsilon_{sw} q_{sx} q_{sw}^{-1} R_{sx,sw} \\ &= (-\varepsilon_x)(-\varepsilon_w) q_x q_w^{-1} q_w^{-1} q R_{sx,sw} \\ &= \varepsilon_x \varepsilon_w q_x q_w^{-1} R_{x,w}\end{aligned}$$

Second case is when $sx > x$, where we have $R_{x,w} = (q-1)R_{x,sw} + qR_{sx,sw}$, which implies

$$\begin{aligned}\bar{R}_{x,w} &= -q^{-1}(q-1)\bar{R}_{x,sw} + q^{-1}\bar{R}_{sx,sw} \\ &= -q^{-1}(q-1)\varepsilon_x\varepsilon_{sw}q_xq_{sw}^{-1}R_{x,sw} + q^{-1}\varepsilon_{sx}\varepsilon_{sw}q_{sx}q_{sw}^{-1}R_{sx,sw} \\ &= \varepsilon_x\varepsilon_wq_xq_w^{-1}((q-1)R_{x,sw} + qR_{sx,sw}) \\ &= \varepsilon_x\varepsilon_wq_xq_w^{-1}R_{x,w}\end{aligned}$$

(b) Since $(T_{w^{-1}})^{-1} = \varepsilon_wq_w^{-1}\sum_{x \leq w} \varepsilon_xR_{x,w}T_x$, part (a) gives us:

$$(T_{w^{-1}})^{-1} = \sum_{x \leq w} q_x^{-1}\bar{R}_{x,w}T_x$$

(c) Apply ι to (b), we get

$$T_w = \sum_{y \leq w} q_yR_{y,w}(T_{y^{-1}})^{-1}$$

If we expand the last term we get double summation as

$$T_w = \sum_{y \leq w} q_yR_{y,w} \left(\sum_{x \leq y} \varepsilon_x\varepsilon_yq_y^{-1}R_{x,y}T_x \right)$$

Now compare the coefficients of T_x on both sides of this equation, we get (c).

QED

6 Kazhdan-Lusztig Polynomials

From this section onwards, we use $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ for k and again denote by \mathcal{H} , the resulting algebra. Note that the preceding calculations remain unchanged with this change of ring.

Theorem 3 *For each $w \in W$ there exists unique element $C_w \in \mathcal{H}$ with following properties:*

(a) $\iota(C_w) = C_w$

(b) $C_w = \varepsilon_wq_w^{\frac{1}{2}}\sum_{x \leq w} \varepsilon_xq_x^{-1}\bar{P}_{x,w}T_x$

where $P_{w,w} = 1$ and $P_{x,w} \in \mathbb{Z}[q]$ has degree less than $\frac{1}{2}(l(w) - l(x) - 1)$ if $x < w$. Furthermore, this implies that \mathcal{H} admits basis of $\{C_w\}_{w \in W}$.

PROOF:

As $T_s^{-1} = q^{-1}T_s - (1 - q^{-1})T_1$, we can define $C_s = q^{-\frac{1}{2}}(T_s - qT_1)$.

$$\iota(C_s) = q^{\frac{1}{2}}(q^{-1}T_s - T_1 + q^{-1}T_1 - q^{-1}T_1)$$

which gives $\iota(C_s) = C_s$ and as $P_{s,s} = 1$ and $P_{1,s} = 1$, satisfying requirements of theorem. These polynomials are called *Kazhdan-Lusztig Polynomials*. We begin by assuming the existence of such basis elements and prove their uniqueness.

Uniqueness: By induction on $l(w)$. We have already proved the base case. Assume that the C_v are chosen uniquely satisfying conditions of the theorem, for every $v \in W$, with $l(v) < l(w)$. To show uniqueness of C_w it suffices to show that each of the polynomials $P_{x,w}$ are chosen uniquely. This we will show by induction on $l(w) - l(x)$, as we know that $P_{w,w} = 1$ serves as base case. Now assume that $P_{y,w}$ are selected in a unique way for every y such that $x < y \leq w$. Since $\iota(C_w) = C_w$, we get

$$\sum_{x \leq w} \varepsilon_x \varepsilon_w q^{\frac{1}{2}} q_x^{-1} \bar{P}_{x,w} T_x = \sum_{y \leq w} \varepsilon_y \varepsilon_w q w^{-\frac{1}{2}} q_y P_{y,w} (T_{y^{-1}})^{-1}$$

the last term can be expanded to give the following:

$$\sum_{x \leq w} \varepsilon_x \varepsilon_w q^{\frac{1}{2}} q_x^{-1} \bar{P}_{x,w} T_x = \sum_{y \leq w} \varepsilon_y \varepsilon_w q w^{-\frac{1}{2}} q_y P_{y,w} \left(\sum_{x \leq y} q_x^{-1} \bar{R}_{x,y} T_x \right)$$

Now comparing coefficient of T_x on both sides yields,

$$\varepsilon_x q^{\frac{1}{2}} \bar{P}_{x,w} = \sum_{x \leq y \leq w} \varepsilon_y q w^{-\frac{1}{2}} q_y P_{y,w} \bar{R}_{x,y}$$

This implies,

$$\varepsilon_x q^{\frac{1}{2}} \left(q^{\frac{1}{2}} q_x^{-\frac{1}{2}} \bar{P}_{x,w} - q w^{-\frac{1}{2}} q_x^{\frac{1}{2}} P_{x,w} \right) = \sum_{x < y \leq w} \varepsilon_y q w^{-\frac{1}{2}} q_y P_{y,w} \bar{R}_{x,y} \quad (14)$$

where by induction hypothesis, right hand side is fixed. Moreover, there are no cancellations on left hand side because first term is polynomial in $q^{\frac{1}{2}}$ without constant term, while the other is polynomial in $q^{-\frac{1}{2}}$, again without constant term. This fixes choice of $P_{x,w}$.

Existence: For this part we need the following definitions

Definition 1 For any $z, v \in W$, we say that $z \prec v$ if $z \leq v$ and $P_{z,v}$ is of degree $\frac{1}{2}(l(v) - l(z) - 1)$. In this case, we define $\mu(z, v)$ to be coefficient of $q^{\frac{1}{2}(l(v) - l(z) - 1)}$ in $P_{z,v}$.

To show existence of basis elements C_w satisfying conditions of theorem, we again use induction on $l(w)$. Given $w \in W$, choose $s \in S$ such that $sw < w$ and substitute $v = sw$. By induction hypothesis, above definition makes sense for every y, z such that $l(y) < l(w)$ and $l(z) < l(w)$. We define C_w as

$$C_w = C_s C_v - \sum_{\substack{z \prec v \\ sz < z}} \mu(z, v) C_z \quad (15)$$

Clearly induction hypothesis implies that C_w is ι -invariant. If we expand C_z for every $z \leq w$ on right side of equation we get

$$\bar{C}_w = (q^{-\frac{1}{2}}T_s - q^{\frac{1}{2}}T_1) \sum_{y \leq v} \varepsilon_y \varepsilon_v q^{\frac{1}{2}} q_y^{-1} \bar{P}_{y,v} T_y - \sum_{\substack{z \prec v \\ sz < z}} \mu(z, v) \left(\sum_{y \leq z} \varepsilon_y \varepsilon_z q^{\frac{1}{2}} q_y^{-1} \bar{P}_{y,z} T_y \right)$$

which we can write as given in statement of theorem, if we define $P_{x,w}$ as (we skip the actual calculations to write two separate cases in one equation)

$$P_{x,w} = q^{1-c} P_{sx,sw} + q^c P_{x,sw} - \sum_{\substack{x \leq z \prec sw \\ sz < z}} \mu(z, v) q_z^{-\frac{1}{2}} q_w^{\frac{1}{2}} P_{x,z}$$

where $c = 1$ for $sx < x$ and $c = 0$ for $sx > x$. It is routine to check that maximum degree of $P_{x,w}$ is less than $\frac{1}{2}(l(w) - l(x) - 1)$.

QED

7 Multiplication Formulae

In this section, we will see how \mathcal{H} acts on itself by computing $T_s.C_w$.

Proposition 3 *Let $s \in S$ and $w \in W$. Then we have the following identities:*

- (a) *If $sw < w$, then $T_s.C_w = -C_w$.*
- (b) *If $sw > w$, then*

$$T_s.C_w = qC_w + q^{\frac{1}{2}}C_{sw} + q^{\frac{1}{2}} \sum_{\substack{z \prec w \\ sz < z}} \mu(z, w)C_z$$

PROOF:

First we compute $T_s.C_w$, when $sw > w$. In this case, we use equation (15) to assert

$$\begin{aligned} C_{sw} &= C_s C_w - \sum_z \mu(z, w) C_z \\ &= (q^{-\frac{1}{2}}T_s - q^{\frac{1}{2}}T_1)C_w - \sum_z \mu(z, w)C_z \end{aligned}$$

where summation is over z such that $z \prec w$ and $sz < z$. This implies, by rearranging terms,

$$T_s.C_w = qC_w + q^{\frac{1}{2}}C_{sw} + q^{\frac{1}{2}} \sum_{\substack{z \prec w \\ sz < z}} \mu(z, w)C_z$$

which proves part (b) of proposition.

For the first part assume that $sw < w$. If $l(w) = 1$, we must have $w = s$.

$$\begin{aligned} T_s.C_s &= T_s(q^{-\frac{1}{2}}T_s - q^{\frac{1}{2}}T_1) \\ &= q^{-\frac{1}{2}}((q-1)T_s + qT_1) - q^{\frac{1}{2}}T_s \\ &= -q^{-\frac{1}{2}}T_s + q^{\frac{1}{2}}T_1 \\ &= -C_s \end{aligned}$$

For any $w \in W$ such that $sw < w$, second part applies to sw , which gives

$$\begin{aligned} T_s.C_{sw} &= qC_{sw} + q^{\frac{1}{2}}C_w + q^{\frac{1}{2}} \sum_{\substack{z \prec w \\ sz < z}} \mu(z, w)C_z \\ \Rightarrow C_w &= q^{-\frac{1}{2}}T_sC_{sw} - q^{\frac{1}{2}}C_{sw} - \sum_{\substack{z \prec w \\ sz < z}} \mu(z, w)C_z \end{aligned}$$

Multiplying both sides on left by T_s gives

$$\begin{aligned} T_s.C_w &= q^{-\frac{1}{2}}((q-1)T_s + qT_1)C_{sw} - q^{\frac{1}{2}}T_sC_{sw} + \sum_z \mu(z, w)C_z \\ &= -q^{-\frac{1}{2}}T_sC_{sw} + q^{\frac{1}{2}}C_{sw} + \sum_z \mu(z, w)C_z \\ &= -C_w \end{aligned}$$

where summation is over z such that $z \prec w$ and $sz < z$. This completes the proof of proposition.

QED

Corollary 1 *Let $x < w$. If $sw < w$, but $sx > x$ for some $s \in S$, then $P_{x,w} = P_{sx,w}$.*

PROOF:

From first part of previous proposition, we have $T_sC_w = -C_w$. This implies

$$\sum_{x \leq w} \varepsilon_x q_x^{-1} \bar{P}_{x,w} T_s.T_x = - \sum_{x \leq w} \varepsilon_x q_x^{-1} \bar{P}_{x,w} T_x$$

Comparing coefficient of T_{sx} on both sides gives (using $sx > x$ and both x, sx are less than w).

$$\begin{aligned} \varepsilon_x q_x^{-1} \bar{P}_{x,w} + (q-1)\varepsilon_{sx} q_{sx}^{-1} \bar{P}_{sx,w} &= -\varepsilon_{sx} q_{sx}^{-1} \bar{P}_{sx,w} \\ \Rightarrow \bar{P}_{x,w} &= -q\varepsilon_{sx} \varepsilon_x q_x q_{sx}^{-1} \bar{P}_{sx,w} \end{aligned}$$

which is equivalent to $P_{x,w} = P_{sx,w}$.

QED

8 W -graphs

8.1 Abstract W -graphs

Definition 2 An undirected graph $\Gamma = (V, E)$ together with following data:

- (i) $I(v) \subset S$, for every $v \in V$.
- (ii) $\mu : E \longrightarrow \mathbb{Z} \setminus \{0\}$

is called W -graph, if for free $\mathbb{Z}[q^{\frac{1}{2}}q^{-\frac{1}{2}}]$ -module over V , denoted by \mathcal{A} , (we will use k for

the ring $\mathbb{Z}[q^{\frac{1}{2}}q^{-\frac{1}{2}}]$) following conditions are satisfied:

(a) For each $s \in S$, τ_s defined as

$$\begin{aligned}\tau_s(v) &= -v, \text{ if } s \in I(v) \\ \tau_s(v) &= qv + q^{\frac{1}{2}} \sum_{\substack{(z,v) \in E \\ s \in I(z)}} \mu(z,v)z, \text{ if } s \notin I(v)\end{aligned}$$

is an element of $\text{End}_k(\mathcal{A})$.

(b) If $s, t \in S$ are such that $m = m(s, t) < \infty$, then

$$\underbrace{\tau_s \tau_t \dots}_{m \text{ terms}} = \underbrace{\tau_t \tau_s \dots}_{m \text{ terms}}$$

The two conditions stated in definition ensure a unique representation $\rho : \mathcal{H} \rightarrow \text{End}(\mathcal{A})$, which maps each T_s to τ_s . To show this we only need to check the following

$$\tau_s^2 = (q - 1)\tau_s + q\tau_1$$

where, τ_1 is identity map. This together with condition (b) and the fact that following relations form presentation of \mathcal{H} , give required map.

$$\begin{aligned}T_s^2 &= (q - 1)T_s + aT_1 \\ \underbrace{T_s T_t \dots}_{m(s,t) \text{ terms}} &= \underbrace{T_t T_s \dots}_{m(s,t) \text{ terms}}, \quad m(s, t) < \infty\end{aligned}$$

If $s \in I(v)$, then $\tau_s^2(v) = v$ and $((q - 1)\tau_s + q\tau_1)(v) = -(q - 1)v + qv = v$. Now suppose that $s \notin I(v)$.

$$\tau_s^2(v) = q\tau_s(v) + q^{\frac{1}{2}} \sum_{\substack{(z,v) \in E \\ s \in I(z)}} \mu(z,v)\tau_s(z)$$

Since $s \in I(z)$ for every z in last part of above equation we get the following (where summation is over z , such that $(z, v) \in E$ and $z \in I(z)$).

$$\begin{aligned}\tau_s^2(v) &= q\tau_s(v) - q^{\frac{1}{2}} \sum \mu(z,v)z \\ &= q\tau_s(v) - \tau_s(v) + qv \\ &= ((q - 1)\tau_s + q\tau_1)(v)\end{aligned}$$

This proves the existence of required representation. Now we define ordering on set of vertices in a W -graph.

Definition 3 Let $v, w \in V$. We say that $v \leq_{\Gamma} w$ if there exists a path $v = v_0, v_1, \dots, v_n = w$, such that

- (a) $(v_i, v_{i+1}) \in E$, for every i , $0 \leq i \leq n - 1$
- (b) $I(v_i) \not\subset I(v_{i+1})$, for every i , $0 \leq i \leq n - 1$. Two vertices, $v, w \in V$ are called Γ -equivalent, denoted by $v \sim_{\Gamma} w$ if $v \leq_{\Gamma} w$ and $w \leq_{\Gamma} v$.

We will two preliminary lemmas about abstract W -graphs, before giving explicit Kazhdan-Lusztig Construction of W -graph, decomposing it into left, right and two-sided cells.

Lemma 2 *Let $\Gamma = (V, E, (I(v))_{v \in V}, \mu)$ be a W -graph. For any subset $X \subseteq S$, the restricted graph $\Gamma_X = (V, E, (I(v) \cap X)_{v \in V}, \mu)$ is W_X -graph. Here W_X is subgroup of W generated by X .*

PROOF:

First note that (W_X, X) is a Coxeter System. Let \mathcal{A}_X be free k -module on V , which is same as \mathcal{A} . Let τ_s^X be maps corresponding to Γ_X as defined in Definition 2. Note that for every $s \in X$, $\tau_s^X = \tau_s$, since for any $s \in X$, $s \in I(v) \cap X \iff s \in I(v)$. Therefore, the conditions in the definition are trivially verified.

QED

Lemma 3 *Let $\Gamma = (V, E, (I(v))_{v \in V}, \mu)$ be a W -graph. Let $C \subset V$ be an equivalence class of Γ -equivalent elements. Then induced graph $\Gamma(C) = (C, E \cap (C \times C), (I(v))_{v \in C}, \mu)$ is again a W -graph.*

PROOF:

Let \mathcal{A}_C be free k -module on C . For every $s \in S$, we define following maps on \mathcal{A}_C similar to that in Definition 2.

$$\begin{aligned} \tau'_s(v) &= -v, \text{ if } s \in I(v) \\ \tau'_s(v) &= qv + q^{\frac{1}{2}} \sum_{\substack{(z,v) \in E \\ z \in C \\ s \in I(z)}} \mu(z, v)z \end{aligned}$$

First condition of Definition 2 is trivially verified, since Γ is W -graph. Therefore, $\tau'_s \in \text{End}_k(\mathcal{A})$. To prove the validity of second condition, we need to prove the following assertion:

For $s_1, \dots, s_n \in S$ and $x, z \in C$, the following equality holds:

(E) Coefficient of z in $(\tau_{s_1} \dots \tau_{s_n})(x) = \text{Coefficient of } z \text{ in } (\tau'_{s_1} \dots \tau'_{s_n})(x)$.

Assuming the validity of above condition (E), if $s, t \in S$, such that $m(s, t) < \infty$, then

$$\underbrace{\tau'_s \tau'_t \dots}_{m(s,t) \text{ times}} = \underbrace{\tau'_t \tau'_s \dots}_{m(s,t) \text{ times}}$$

on \mathcal{A}_C , since, for every $x, z \in C$, coefficients of z in both $(\underbrace{\tau'_s \tau'_t \dots}_{m(s,t) \text{ times}})(x)$ and $(\underbrace{\tau'_t \tau'_s \dots}_{m(s,t) \text{ times}})(x)$ are same as those in $(\underbrace{\tau_s \tau_t \dots}_{m(s,t) \text{ times}})(x)$ and $(\underbrace{\tau_t \tau_s \dots}_{m(s,t) \text{ times}})(x)$ respectively using (E), which are same from hypothesis that Γ is W -graph.

To prove condition E , we use induction on n . First note that for any $x \in V$ and $s \in S$,

coefficient of $z \in V$ is non-zero in $\tau_s(x)$ implies that $z \leq_\Gamma x$ (because either $z = x$ or $s \in I(z)$ and $s \notin I(x)$, which means $I(z) \not\subseteq I(x)$ and since $(z, x) \in E$, we get that $z \leq_\Gamma x$). For the base case, i.e, $n = 1$, the assertion is trivially verified. Assume $n > 1$ and (E) holds for every $m < n$. Let $s_1, \dots, s_n \in S$ and $x \in C$. Apply induction hypothesis to s_2, \dots, s_n to write

$$\begin{aligned} (\tau_{s_2} \dots \tau_{s_n})(x) &= \sum_{z \in C} n_z z + \sum_{v \in V \setminus C} n_v v \\ (\tau'_{s_2} \dots \tau'_{s_n})(x) &= \sum_{z \in C} n_z z \end{aligned}$$

For any $z, v \in V$, if we denote by $n_{v,z}$, the coefficient of z in $\tau_{s_1}(x)$, then assuming $z \in C$, the coefficient of z in $(\tau_{s_1} \dots \tau_{s_n})(x)$ is same as

$$\sum_{v \in V} n_{v,z} n_v = \sum_{v \in C} n_{v,z} n_v$$

since for any $v \in V$, such that $n_v \neq 0$, we have $v \leq_\Gamma x$. If besides this $n_{v,z} \neq 0$, we get $z \leq_\Gamma v$ and since $x \sim_\Gamma z$, we conclude that $v \in C$.

The later term is same as coefficient of z in $(\tau'_{s_1} \dots \tau'_{s_n})(x)$, which proves the condition (E) and hence completes proof of the lemma.

QED

8.2 Kazhdan-Lusztig Construction

In this section, we give one explicit construction for W -graphs and definitions of left cells. Similarly, one can define right cells and two sided cells, which we will omit.

Construct W -graph Γ , as follows. Let $V = W$ be set of vertices. For every $w \in W$, let

$$I_l(w) := \{s \in S : sw < w\}$$

Let edge set be given be $E = \{(v, w) : v, w \in W, v \prec w\}$.

For $(v, w) \in E$, the function μ is defined similar to function in Definition 1. That is, $\mu(v, w) = \mu(w, v)$. Exactly, one of the previous two terms is given in Definition 1.

Theorem 4 *The graph constructed above Γ is a W -graph.*

PROOF:

If \mathcal{A} denote free algebra on set of vertices of Γ , which is same as W , then from Theorem 3, we can denote basis of \mathcal{A} by $\{C_w\}_{w \in W}$. We will prove that the maps τ_s , defined in Definition 2, are same as left multiplication by T_s in Hecke algebra \mathcal{H} of (W, S) .

Observing the following two relations, proved in Proposition 3,

$$T_s.C_w = -C_w, \text{ if } sw < w$$

$$T_s.C_w = qC_w + q^{\frac{1}{2}}C_{sw} + q^{\frac{1}{2}} \sum_{\substack{z \prec w \\ sz < z}} \mu(z, w)C_z, \text{ if } sw > w$$

we can conclude that it suffices to prove that if $sw > w$, then only z , such that $sz < z$ and $w \prec z$, is $z = sw$. That is we have to prove that if $sw > w$, $w < z$, $sz < z$ then

$$w \prec z \iff z = sw$$

Moreover, in this case $\mu(w, z) = 1$. Use the Corollary 1 to conclude that in this situation, $P_{w,z} = P_{sw,z}$. If $z \neq sw$, then we get

$$\deg(P_{w,z}) = \deg(P_{sw,z}) = \frac{1}{2}(l(z) - l(w) - 1 - 1) < \frac{1}{2}(l(z) - l(w) - 1)$$

which implies that $w \not\prec z$. If $z = sw$, then $P_{w,z} = P_{z,z} = 1$, which implies that $\mu(w, z) = 1$. This completes proof of theorem.

QED

An equivalence class under relation \sim_Γ is called left cell of W . If we define

$$I_r(w) := \{s \in S : ws < w\}$$

we get right cells. In order to get two sided cells of W , we can consider $(W \times W^o, S \cup S^o)$ as Coxeter System. Then we define $I(w) := I_l(w) \cup I_r(w)^o$.

9 Hecke algebra of (G, B) pair

In this section, we define Hecke algebra for a pair (G, B) , where B is subgroup of group G , such that equivalent conditions of following proposition are satisfied:

Proposition 4 *Let G be a group and B be its subgroup. Then the following are equivalent:*

- (a) *For any $g \in G$, double coset BgB is finite union of q_g left B -cosets in G .*
- (b) *$q_g = [B : B \cap gBg^{-1}] < \infty$, for every $g \in G$.*

Also, in this case $q_{gh} \leq q_g q_h$.

We assume throughout that above equivalent conditions hold for pair (G, B) under consideration.

Theorem 5 *Let k be commutative ring with identity $1 \neq 0$. For every $s \in G/B$ (resp. $t \in B \setminus G/B$), we denote by a_s and b_t , the following maps from G to k :*

$$a_s(g) = \begin{cases} 1, & g \in s \\ 0, & g \notin s \end{cases}$$

$$b_t(g) = \begin{cases} 1, & g \in t \\ 0, & g \notin t \end{cases}$$

Let L be k -module generated by $\{a_s : s \in G/B\}$ and H be k -module generated by $\{b_t : t \in B \setminus G/B\}$. Then

- (a) There exists a unique linear form μ on L , such that $\mu(a_s) = 1$, for every $s \in G/B$.
- (b) For $\varphi \in L$, $\psi \in H$ and $x \in G$, the following map from G to k ,

$$\theta_x(\varphi, \psi) : y \longmapsto \varphi(y)\psi(y^{-1}x)$$

is an element of L .

- (c) Define the following map for any $\varphi \in L$ and $\psi \in H$

$$\varphi * \psi : x \longmapsto \mu(\theta_x(\varphi, \psi))$$

This map from G to k belongs to L . Moreover, if $\varphi \in H$, then $\varphi * \psi \in H$. Thus

$$(\varphi, \psi) \longmapsto \varphi * \psi$$

gives structure of k -algebra on H , called Hecke algebra of G with respect to B , denoted by $H_k(G, B)$. And under the same map, L becomes right H -module.

Theorem 6 Assume that (G, B, N, S) is a Tits System with associated Coxeter system (W, S) . Further suppose that for every $s \in S$, $C(s)$ is finite union of q_s left B -cosets in G . Then conditions of Proposition 4 hold. Let $H_k(G, B)$ be Hecke algebra over k of G with respect to B (defined in Theorem 5). Denote by $b_w = b_{C(w)}$, adhering to notations of Theorem 5, which form basis of $H_k(G, B)$.

- (a) For $s \in S$ and $w \in W$, the following relations hold:

$$b_s * b_s = (q_s - 1)b_s + q_s$$

$$b_s * b_w = b_{sw}, \text{ if } sw > w$$

- (b) Consider the generic algebra $\mathcal{E}_k((q_s - 1), q_s)$ on (W, S) . The map $T_w \longmapsto b_w$ is then isomorphism of k -algebras.

10 Closed form expression for R -polynomials

To state the formula for computing R -polynomials, we need to first fix some notations and definitions. Let (W, S) be a Coxeter System and let $w \in W$. Fix $w = s_1 \dots s_p$ reduced expression for w . For any $x \in W$, we say $x \leq w$ in Bruhat-Chevalley ordering on W , if there exists i_1, \dots, i_k , such that

$$x = s_1 \dots s_{i_1}^{\hat{}} \dots s_{i_k}^{\hat{}} \dots s_p$$

Definition 4 By a subexpression of (s_1, \dots, s_p) , we mean a sequence $\underline{\sigma} = (\sigma_0, \dots, \sigma_p)$, where $\sigma_0 = 1$ and for each i , $1 \leq i \leq p$, $\sigma_i = \sigma_{i-1}$ or $\sigma_{i-1}s_i$

Let \mathcal{S} denote the set of subexpressions of (s_1, \dots, s_p) . We fix the following notations for any $x \leq w$ and any $\underline{\sigma} \in \mathcal{S}$.

$$\mathcal{S}(x) = \{\underline{\sigma} \in \mathcal{S} : \sigma_p = x\}$$

$$I(\underline{\sigma}) = \{j : \sigma_{j-1} = \sigma_j\}$$

$$J(\underline{\sigma}) = \{j : \sigma_j < \sigma_{j-1}\}$$

$$n(\underline{\sigma}) = |I(\underline{\sigma})| ; m(\underline{\sigma}) = |J(\underline{\sigma})|$$

Also define a map $\pi : \mathcal{S} \longrightarrow W$ by $\pi(\underline{\sigma}) = \sigma_p$.

Definition 5 An element $\underline{\sigma} \in \mathcal{S}$ is called distinguished if for every j , $1 \leq j \leq p$, we have $\sigma_j \leq \sigma_{j-1}s_j$

We denote by \mathcal{D} , the set of all distinguished subexpressions and $\mathcal{D}(x) = \mathcal{D} \cap \mathcal{S}(x)$.

Theorem 7 For $x \leq w$, x, w as above, we have

$$R_{x,w}(q) = \sum_{\underline{\sigma} \in \mathcal{D}(x)} q^{m(\underline{\sigma})} (q-1)^{n(\underline{\sigma})} \quad (16)$$

The proof of this theorem will take up rest part of this section. First we observe that for any $\underline{\sigma} \in \mathcal{S}$, we have

$$l(\pi(\underline{\sigma})) = p - n(\underline{\sigma}) - 2m(\underline{\sigma})$$

For any $i \in \mathbb{Z}^+$, define

$$\mathcal{S}_i(x) = \{\underline{\sigma} \in \mathcal{S}(x) : m(\underline{\sigma}) = i\}$$

$$\mathcal{D}_i(x) = \{\underline{\sigma} \in \mathcal{D}(x) : m(\underline{\sigma}) = i\}$$

Let $\mathcal{S}', \mathcal{D}', \dots$ be corresponding sets for $ws_p = s_1 \dots s_{p-1}$. Define

$$\begin{aligned} \theta : \mathcal{S} &\longrightarrow \mathcal{S}' \\ (\sigma_0, \dots, \sigma_p) &\longmapsto (\sigma_0, \dots, \sigma_{p-1}) \end{aligned}$$

It is clear that θ maps \mathcal{D} to \mathcal{D}' .

Lemma 4 *Assume $xs_p \leq x$. Then,*

1. $xs_p \leq ws_p$
2. $p \notin I(\underline{\sigma}), \forall \underline{\sigma} \in \mathcal{D}(x)$ and $\theta(\mathcal{D}(x)) \subseteq \mathcal{D}'(xs_p)$
3. $n(\underline{\sigma}) = n(\theta(\underline{\sigma})), \forall \underline{\sigma} \in \mathcal{D}(x)$ and $\theta(\mathcal{D}_i(x)) \subseteq \mathcal{D}'_i(xs_p), \forall i$
4. $\theta : \mathcal{D}_i(x) \longrightarrow \mathcal{D}'_i(xs_p)$ is bijective map.

PROOF:

We begin with one elementary observation. Assume $x \leq w$ and $x \not\leq ws_p$. In this case x can be obtained from reduced expression $w = s_1 \dots s_p$, but not from $ws_p = s_1 \dots s_{p-1}$. Therefore, xs_p can be obtained from ws_p and hence $xs_p \leq ws_p$. Now the proof of first part is trivial since $x \leq ws_p$ also implies $xs_p \leq ws_p$.

For second part, assume the contrary, i.e, $p \in I(\underline{\sigma})$ for some distinguished subexpression $\underline{\sigma} \in \mathcal{D}(x)$. This means $\sigma_{p-1} = \sigma_p$, which because of definition of distinguished subexpression implies that $\sigma_{p-1}s_p > \sigma_p$ and hence $xs_p > x$, which is contradiction.

Therefore, we certainly have $\sigma_p = \sigma_{p-1}s_p$, for any distinguished subexpression $\underline{\sigma}$ and hence $\pi(\theta(\underline{\sigma})) = \sigma_{p-1} = xs_p$ which gives $\theta(\mathcal{D}(x)) \subseteq \mathcal{D}'(xs_p)$. And $n(\theta(\underline{\sigma})) = n(\underline{\sigma})$. This together with $l(x) - 1 = l(xs_p) = p - 1 - n(\theta(\underline{\sigma})) - 2m(\theta(\underline{\sigma}))$ proves that $m(\underline{\sigma}) = m(\theta(\underline{\sigma}))$ and hence we have $\theta(\mathcal{D}_i(x)) \subseteq \mathcal{D}'_i(xs_p)$ for each i .

To prove that $\theta : \mathcal{D}_i(x) \longrightarrow \mathcal{D}'_i(xs_p)$ is bijective map, we produce inverse map as $\phi((\sigma_0, \dots, \sigma_{p-1})) = (\sigma_0, \dots, \sigma_{p-1}, \sigma_{p-1}s_p)$. From previous discussion, it is now clear that this map is well defined inverse of θ and hence θ restricted to $\mathcal{D}_i(x)$ maps it bijectively to $\mathcal{D}'_i(xs_p)$.

QED

Lemma 5 *Assume $xs_p \geq x$ and $xs_p \not\leq w$, then*

1. $x \leq ws_p$
2. $p \in I(\underline{\sigma}), \forall \underline{\sigma} \in \mathcal{D}(x)$ and $\theta(\mathcal{D}(x)) \subseteq \mathcal{D}'(x)$
3. $n(\underline{\sigma}) = n(\theta(\underline{\sigma})) + 1, \forall \underline{\sigma} \in \mathcal{D}(x)$ and moreover, $\theta(\mathcal{D}_i(x)) \subseteq \mathcal{D}'_i(x), \forall i$.
4. $\theta : \mathcal{D}_i(x) \longrightarrow \mathcal{D}'_i(x)$ is bijective map.

PROOF:

The proof of first part is contained in proof of previous lemma. Assume, if possible, $x \not\leq ws_p$. This together with initial assumption that $x \leq w$ gives $xs_p \leq ws_p$ which is against hypothesis of lemma.

Now again assume the contrary that $p \notin I(\underline{\sigma})$ for some $\underline{\sigma} \in \mathcal{D}(x)$. This means $\sigma_p = \sigma_{p-1}s_p$, that is, $\sigma_{p-1} = xs_p$. Hence, xs_p can be obtained as subexpression of ws_p and hence $xs_p \leq ws_p$, which is contradiction. This proves that $\pi(\theta(\underline{\sigma})) = \sigma_{p-1} = x$ and hence $\theta(\mathcal{D}(x)) \subseteq \mathcal{D}'(x)$.

Since $\sigma_{p-1} = \sigma_p$ for any distinguished subexpression $\underline{\sigma}$ for x , we have $n(\underline{\sigma}) = n(\theta(\underline{\sigma})) + 1$. This together with the relations $l(x) = p - n(\underline{\sigma}) - 2m(\underline{\sigma}) = p - 1 - n(\theta(\underline{\sigma})) - 2m(\theta(\underline{\sigma}))$, proves that $m(\underline{\sigma}) = m(\theta(\underline{\sigma}))$ and hence $\theta(\mathcal{D}_i(x)) \subseteq \mathcal{D}'(x)$.

To prove that $\theta : \mathcal{D}_i(x) \longrightarrow \mathcal{D}'(x)$ is bijective, we give inverse map as $\phi((\sigma_0, \dots, \sigma_{p-1})) = (\sigma_0, \dots, \sigma_{p-1}, \sigma_{p-1})$ and from previous discussion it is clear that this is inverse to θ and hence θ maps $\mathcal{D}_i(x)$ bijectively to $\mathcal{D}'_i(x)$.

QED

Lemma 6 Assume $x \leq xs_p \leq ws_p$, then define the following two sets:

$$\mathcal{A}(x) = \{\underline{\sigma} \in \mathcal{D}(x) : p \notin I(\underline{\sigma})\}$$

$$\mathcal{B}(x) = \mathcal{D}(x) \setminus \mathcal{A}(x)$$

1. $n(\underline{\sigma}) = n(\theta(\underline{\sigma}))$, $\forall \underline{\sigma} \in \mathcal{A}(x)$ and $\theta(\mathcal{A}(x) \cap \mathcal{D}_i(x)) \subseteq \mathcal{D}'_{i-1}(xs_p)$
2. $\theta : \mathcal{A}(x) \cap \mathcal{D}_i(x) \longrightarrow \mathcal{D}'_{i-1}(xs_p)$ is a bijective map.
3. $n(\underline{\sigma}) = n(\theta(\underline{\sigma})) + 1$, $\forall \underline{\sigma} \in \mathcal{B}(x)$ and $\theta(\mathcal{B}(x) \cap \mathcal{D}_i(x)) \subseteq \mathcal{D}'_i(x)$
4. $\theta : \mathcal{B}(x) \cap \mathcal{D}_i(x) \longrightarrow \mathcal{D}'_i(x)$ is a bijective map.

The proof of this lemma is entirely contained in previous two lemmas.

Now we proceed with proof of main theorem (7). We prove the formula (16) by induction on $l(w)$, base case for $w = 1$ or $w = s$ being clear. We have the following cases:

(A) $xs_p \leq x$. In this case, we know that $R_{x,w}(q) = R_{xs_p,ws_p}(q)$, therefore using induction and Lemma 4, we have the following:

$$\begin{aligned} R_{xs_p,ws_p}(q) &= \sum_{\underline{\sigma}' \in \mathcal{D}'(xs_p)} (q-1)^{n(\underline{\sigma}')} q^{m(\underline{\sigma}')} \\ &= \sum_{\underline{\sigma} \in \mathcal{D}(x)} (q-1)^{n(\theta(\underline{\sigma}))} q^{m(\theta(\underline{\sigma}))} \\ &= \sum_{\underline{\sigma} \in \mathcal{D}(x)} (q-1)^{n(\underline{\sigma})} q^{m(\underline{\sigma})} \end{aligned}$$

(B) $x \leq xs_p$ and $xs_p \not\leq ws_p$. In this case, we know that $R_{x,w}(q) = (q-1)R_{x,ws_p}(q)$. Using Lemma 5, we get the following:

$$\begin{aligned} R_{x,w}(q) &= (q-1) \sum_{\underline{\sigma}' \in \mathcal{D}'(x)} (q-1)^{n(\underline{\sigma}')} q^{m(\underline{\sigma}')} \\ &= (q-1) \sum_{\underline{\sigma} \in \mathcal{D}(x)} (q-1)^{n(\theta(\underline{\sigma}))} q^{m(\theta(\underline{\sigma}))} \\ &= (q-1) \sum_{\underline{\sigma} \in \mathcal{D}(x)} (q-1)^{n(\underline{\sigma})-1} q^{m(\underline{\sigma})} \end{aligned}$$

(C) $x \leq xs_p$ and $xs_p \leq ws_p$. In this case, we know that $R_{x,w}(q) = (q-1)R_{x,ws_p}(q) + qR_{xs_p,ws_p}(q)$. We compute both these terms separately, using Lemma 6. Firstly, $R_{x,ws_p}(q) = \sum_{\underline{\sigma}' \in \mathcal{D}'(x)} (q-1)^{n(\underline{\sigma}')} q^{m(\underline{\sigma}')}$. The later term can be written as

$$R_{x,ws_p}(q) = \sum_i \left(\sum_{\underline{\sigma}' \in \mathcal{D}'_i(x)} (q-1)^{n(\underline{\sigma}')} \right) q^i$$

Since $\mathcal{D}'_i(x)$ is in bijection with $\mathcal{B}(x) \cap \mathcal{D}_i(x)$ under the map θ , and $n(\theta(\underline{\sigma})) = n(\underline{\sigma}) - 1$ we write it again as

$$R_{x,ws_p}(q) = \sum_i \left(\sum_{\underline{\sigma} \in \mathcal{B}(x) \cap \mathcal{D}_i(x)} (q-1)^{n(\underline{\sigma})-1} \right) q^i$$

which is obviously equal to $\sum_{\underline{\sigma} \in \mathcal{B}(x)} (q-1)^{n(\underline{\sigma})-1} q^{m(\underline{\sigma})}$.

Similarly, the second term becomes

$$R_{xs_p,ws_p}(q) = \sum_{\underline{\sigma} \in \mathcal{A}(x)} (q-1)^{n(\underline{\sigma})} q^{m(\underline{\sigma})-1}$$

Substituting these in expression for $R_{x,w}(q)$, we get (16), for this case too.

References

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