Kazhdan Lusztig Cells in Permutation Groups

Sachin Gautam

Throughout these lectures we use the following notations:

$$[n] = \{1, \dots, n\}$$

$\mathfrak{S}_n$ denotes permutation group of $[n]$. Elements of $\mathfrak{S}_n$ are denoted interchangably by one line notation and cyclic notation as follows:

$$\pi = i_1 \ \dots \ i_n$$

The above notation called *one line notation* means that $\pi(j) = i_j, \ 1 \leq j \leq n$.

$$\pi = (i_1^1, \dots, i_{k_1}^1)(i_1^2, \dots, i_{k_2}^2) \dots (i_1^r, \dots, i_{k_r}^r)$$

The above notation called *cycle notation* means (for every $1 \leq j \leq r$ and $1 \leq l \leq k_j$)

$$\pi(i_l^j) = \begin{cases} i_{l+1}^j, & if \quad l < k_j \\ i_1^j, & if \quad l = k_j \end{cases}$$

and make the assumption that $\pi(m) = m$, whenever m doesn't appear in above expression. We write s_i for the transposition map $s_i = (i, i+1)$, for each $1 \leq i \leq n-1$.

$$S := \{s_1, \dots, s_{n-1}\} \subset \mathfrak{S}_n$$

Let ϵ denote the identity in $\mathfrak{S}_n$

1 Some general results

Theorem 1 *The pair $(\mathfrak{S}_n, S)$ is Coxeter System. Moreover, for any $\pi \in \mathfrak{S}_n$ and $s_i \in S$, we have the following:*

$$l(s_i\pi) \leq l(\pi) \iff \pi^{-1}(i) > \pi^{-1}(i+1) \tag{1}$$

$$l(\pi s_i) \leq l(\pi) \iff \pi(i) > \pi(i+1) \tag{2}$$

Proof:
For any i, $1 \leq i \leq n-1$, consider the following set

$$H_i := \{\pi \in \mathfrak{S}_n \ni \pi^{-1}(i) < \pi^{-1}(i+1)\}$$

From partition axioms of Coxeter System, it suffices to prove that

(P1) $\epsilon \in H_i,\ \forall\, i,\ 1 \le i \le n-1$
Since $\epsilon^{-1}(i) = i < i+1 = \epsilon^{-1}(i+1)$, $(P1)$ is obviously satisfied.

(P2) For every i, $1 \le i \le n-1$, $H_i \cap s_i H_i = \phi$
If $\pi \in H_i$, then

$$(s_i\pi)^{-1}(i) = \pi^{-1}(i+1) > \pi^{-1}(i) = (s_i\pi)^{-1}(i+1)$$

Therefore, $s_i\pi \notin H_i$, which proves $(P2)$.

(P3) If $\pi \in \mathfrak{S}_n$ and $1 \le i, j \le n-1$, then $\pi \in H_i$ and $\pi s_j \notin H_i$ implies that $s_i\pi = \pi s_j$
Let us denote $l_1 = \pi^{-1}(i)$ and $l_2 = \pi^{-1}(i+1)$. From the hypothesis, we have $l_1 < l_2$. Since $\pi s_j \notin H_i$, we have

$$s_j\pi^{-1}(i) > s_j\pi^{-1}(i+1)$$

or equivalently saying, we have $s_j(l_1) > s_j(l_2)$ and $l_1 < l_2$. It is possible only if $l_1 = j$ and $l_2 = j+1$, in which case, we can trivially verify $(P3)$.

QED

Corollary 1 *For any $\pi \in \mathfrak{S}_n$, we have the following (where $l(w)$ is length function on Coxeter System $(\mathfrak{S}_n, S)$)*

$$l(\pi) = \#\{(i,j) \ni 1 \le i < j \le n,\ \pi(i) > \pi(j)\}$$

Corollary 2 *If we denote by $' \le '$, the Bruhat-Chevalley ordering on $(\mathfrak{S}_n, S)$ and assume the following notations:*
For any sequence of distinct elements $(a_1, \ldots, a_r)$, $[a_1, \ldots, a_r]$ denotes the sequence ordered in increasing fashion. We say $(a_1, \ldots, a_r) \le (b_1, \ldots, b_r)$ iff for every i, $1 \le i \le r$, $a_i \le b_i$. Then the following holds for any two $\pi, \sigma \in \mathfrak{S}_n$

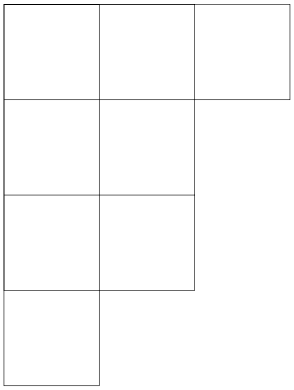
$$\pi \le \sigma \iff \forall k,\ 1 \le k \le n,\ [\pi(1), \ldots, \pi(k)] \le [\sigma(1), \ldots, \sigma(k)]$$

2 Some Combinatorics

1. $\lambda = (\lambda_1, \ldots, \lambda_k)$ is said to be partition of n if

$$\sum_{i=1}^{k} \lambda_i = n \text{ and } \lambda_1 \ge \ldots \ge \lambda_k$$

2. Young Diagram Corresponding to each partition λ of n, we associate array having lengths of columns $\lambda_i's$ (called Young Diagram). For example, following is Young diagram for $\lambda = (4, 3, 1)$



3. <u>Partial Tableaux</u> is obtained by assigning distinct numbers to each cell of Young Diagram, such that each row and each column has increasing entries. If Young diagram is associated with partition λ, the resulting Partial Tableaux is said to be of shape λ. If entries in cells of Young Diagram are $\{1, \dots, n\}$ (where λ is partition of n), the resulting tableux is called *Standard Tableaux.* For example, corresponding to above mentioned Diagram, the following is standard tableaux (of shape $(4, 3, 1)$)

1	4	6
2	7	
3	8	
5		

2.1 Robinson-Schensted Correspondance

The main aim of this section is to describe the bijective correspondance between elements of $\mathfrak{S}_n$ and pair of standard tableaux of same shape

$$\pi \longleftrightarrow (P(\pi), Q(\pi))$$

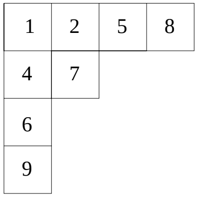
called *Robinson Schensted correspondance*

Step 1: Insertion procedure
Let P_λ be a partial tableaux. We first describe a procedure to insert a number n in P_λ.
$Insert(P_\lambda, n)$

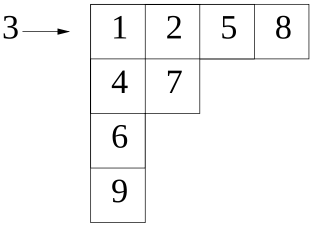
1. Set R to be first row of P_λ.

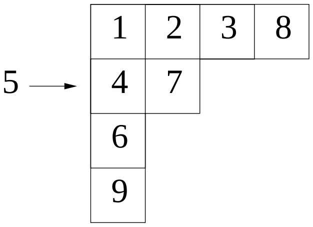
2. If n is greater than all the entries in R, insert n in the end of R and stop.
3. If m is smallest element of R, greater than n, replace m by n and set $n := m$.
4. Repeat previous two steps, setting R to be next row.

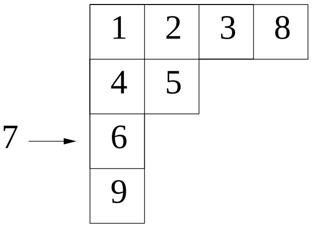
For example, consider inserting $n = 3$, in the following tableaux:

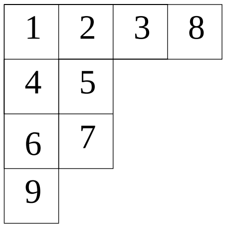


which includes the following steps:









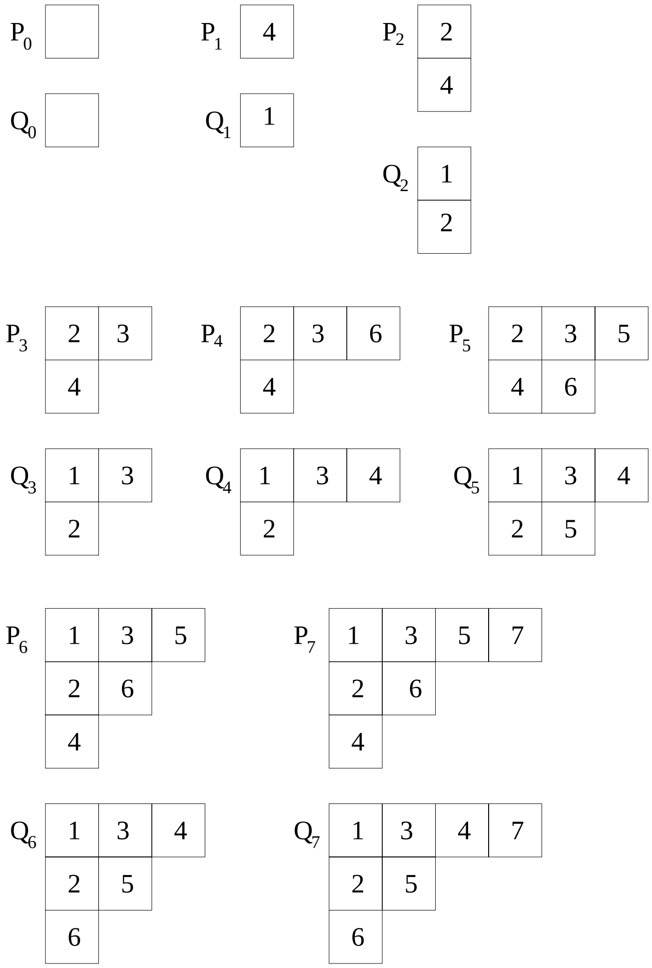
R-S Procedure: To construct $(P(\pi), Q(\pi))$ from $\pi \in \mathfrak{S}_n$.
Let $\pi = x_1 \ \ldots \ x_n$ be given as one line notation. Initialize $P_0 = \phi = Q_0$

1. For $k := 1$ *to* n, do:
2. $P_k = Insert(P_{k-1}, x_k)$
3. If insertion of x_k ended in $(i,j)^{th}$ cell, define Q_k to be obtained from Q_{k-1} by placing k at $(i,j)^{th}$ place.

Ouput $P(\pi) = P_n$ and $Q(\pi) = Q_n$.
For example, consider the following element of $\mathfrak{S}_7$,

$$\pi = 4\ 2\ 3\ 6\ 5\ 1\ 7$$

The above algorithm runs in following steps



Inverse R-S correspondance To show that above procedure gives bijective correspondance between elements of $\mathfrak{S}_n$ and standard tableaux of same shape $\lambda \vdash n$, it is enough to produce inverse map. The following is procedure to produce $\pi \in \mathfrak{S}_n$ given two standard tableaux P and Q of same shape $\lambda \vdash n$. Initialize $P_n = P$

1. For each k, find x_k with the following ($k :=$ *n to* 0)
2. Find (i, j) such that (i, j) cell of Q_k contains k. Let Q_{k-1} be obtained from Q_k by erasing (i, j) cell.
3. Set $x := P_k(i, j)$ and erase $P_k(i, j)$ (call it P_{k-1}). Set R to be $i - 1$ row of P_k.
4. While R is not zeroth row of P_k (assume it to be empty row)
    - Let y be longeset element of R smaller than x. Replace y by x in R.
    - $x := y$ and R be next row upwards.
5. $x_k := x$.

The output is $\pi = x_1 \ \ldots \ x_n$. Take for example the following two standard tableaux with same shape

$$\begin{array}{|c|c|c|c|}\hline 1&3&5&7\\ \hline 2&6 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1&3&4&7\\ \hline 2&5 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}$$

The procedure described above takes the following steps:

$$P_7 \ \begin{array}{|c|c|c|c|}\hline 1&3&5&7\\ \hline 2&6 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array} \qquad P_6 \ \begin{array}{|c|c|c|}\hline 1&3&5\\ \hline 2&6 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}$$

$$Q_7 \ \begin{array}{|c|c|c|c|}\hline 1&3&4&7\\ \hline 2&5 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array} \qquad Q_6 \ \begin{array}{|c|c|c|}\hline 1&3&4\\ \hline 2&5 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}$$

$$P_5 \ \begin{array}{|c|c|c|}\hline 2&3&5\\ \hline 4&6 \\ \cline{1-2}\end{array} \qquad P_4 \ \begin{array}{|c|c|c|}\hline 2&3&6\\ \hline 4 \\ \cline{1-1}\end{array} \qquad P_3 \ \begin{array}{|c|c|}\hline 2&3\\ \hline 4 \\ \cline{1-1}\end{array}$$

$$Q_5 \ \begin{array}{|c|c|c|}\hline 1&3&4\\ \hline 2&5 \\ \cline{1-2}\end{array} \qquad Q_4 \ \begin{array}{|c|c|c|}\hline 1&3&4\\ \hline 2 \\ \cline{1-1}\end{array} \qquad Q_3 \ \begin{array}{|c|c|}\hline 1&3\\ \hline 2 \\ \cline{1-1}\end{array}$$

$$P_2 \ \begin{array}{|c|}\hline 2\\ \hline 4 \\ \hline\end{array} \qquad P_1 \ \begin{array}{|c|}\hline 4\\ \hline\end{array}$$

$$Q_2 \ \begin{array}{|c|}\hline 1\\ \hline 2 \\ \hline\end{array} \qquad Q_1 \ \begin{array}{|c|}\hline 1\\ \hline\end{array}$$

and hence the resulting permutation is given as : $\pi = 4\ 2\ 3\ 6\ 5\ 1\ 7$.

2.2 Knuth Relations

Definition 1 Two elements $\pi, \sigma \in \mathfrak{S}_n$ are said to be P-equivalent if $P(\pi) = P(\sigma)$, written as $\pi \sim^P \sigma$.

Definition 2 Two permutations $\pi, \sigma \in \mathfrak{S}_n$ are said to be related by Knuth relation of

1. *First kind* (written as $\pi \sim^1 \sigma$) if there exist $x < y < z$ such that

   $$\pi = x_1 \ \ldots y \ z \ x \ldots \ x_n \ and \ \sigma = x_1 \ \ldots \ y \ x \ z \ \ldots \ x_n$$

   or vice versa.

2. *Second kind* (written as $\pi \sim^2 \sigma$) if there exist $x < y < z$ such that

   $$\pi = x_1 \ \ldots z \ x \ y \ldots \ x_n \ and \ \sigma = x_1 \ \ldots \ x \ z \ y \ \ldots \ x_n$$

Definition 3 Two permutations $\pi, \sigma \in \mathfrak{S}_n$ are said to be Knuth equivalent $\pi \sim^K \sigma$ if they are linked as

$$\pi = \pi_0 \sim \ldots \pi_n = \sigma$$

where π_i and π_{i+1} are related by Knuth relation of first or second kind.

Definition 4 If P is a tableaux, the *row word* of P is the permutation

$$\pi_P = R_l \ldots R_1$$

where $R_1, \ldots, R_l$ are rows of P.

For example, if P is :

1	3	5	7
2	6		
4			

then in this case $\pi_P = 4\ 2\ 6\ 1\ 3\ 5\ 7$.

Theorem 2 *[Knuth] Let $\pi, \sigma \in \mathfrak{S}_n$. Then*

$$\pi \sim^P \sigma \iff \pi \sim^K \sigma$$

PROOF:
Assume that $\pi \sim^1 \sigma$. That is, there are $x < y < z$ such that π and σ have following form:

$$\pi = \ldots y \ x \ z \ldots$$

$$\sigma = \ldots y\ z\ x \ldots$$

Since all the elements entered before y are same, it suffices to prove that for any partial tableaux P, inserting y, x, z and y, z, x in respective orders yield the same tableaux. If we denote by $r_y(P)$, tableaux resulting from inserting y in P, we have to prove the following

$$r_z r_x r_y(P) = r_x r_z r_y(P) \tag{3}$$

We will prove this claim by induction on number of rows in P. For $P = \phi$, both sides of (3) yield the following tableaux

$$\begin{array}{|c|c|}\hline x & z \\ \hline y & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

Now assume that P has $r > 0$ rows. Suppose y enters first row at k^{th} column, replacing y'. We will examine the operations on both sides of (3) respectively.
Assume x is inserted next. Since $x < y$, x replaces some x' from column j, with $j \leq k$. Furthermore, $x' < y'$. Similarly, $z > y$ implies that z replaces some element z' from column l, with $l > k$ and $z' > y'$.
Considering the right hand side of (3), we get that z and x, if inserted in this respective order replace same elements z' and x' from same columns l and j respectively. Therefore, first rows of two tableaux obtained are same. Moreover, the rest of tableaux is obtained by inserting y', x', z' and y', z', x' in a tableaux of strictly smaller number of rows, in their respective order. Since the same order $x' < y' < z'$, still holds, we can appeal to induction to assert that rest of tableaux are also same. which completes one half of the proof.
If π and σ are related by Knuth relation of second kind, the similar argument will work.

We will show that

$$\pi \sim^K \pi_P$$

where $P(\pi) = P$. Since Knuth relations are transitive, the converse of theorem will follow. We induct on n, number of elements of π. Base case is trivial, since for $n = 1$, $\pi = \pi_P$. Assume that x is last element of π, that is, π is written in one line notation as

$$\pi = \ldots x$$

Or $\pi = \pi' x$, where π' is sequence of $n - 1$ elements. Therefore, by induction we have

$$\pi' \sim^K \pi_{P'}$$

where $P' = P(\pi')$. Thus it suffices to prove that

$$\pi_{P'} x \sim^K \pi_P$$

Let $R_1, \ldots, R_l$ be rows of P' and assume $R_1 = p_1 \ \ldots \ p_k$. If x enters P' in column j, then,

$$p_1 < \ldots p_{j-1} < x < p_j < \ldots p_k$$

Therefore, we have following sequence of Knuth operations

$$\begin{array}{rcl} \pi_{P'}x & = & R_l \ldots R_2 \, p_1 \ \ldots \ p_k \, x \\ & \sim^1 & R_l \ldots R_2 \, p_1 \ \ldots \ p_{k-1} \, x \, p_k \\ & \vdots & \\ & \sim^1 & R_l \ldots R_2 \, p_1 \ \ldots \ p_{j-1} \, p_j \, x \, p_{j+1} \ \ldots \ p_k \\ & \sim^2 & R_l \ldots R_2 \, p_1 \ \ldots \ p_j \, p_{j-1} \, x \, p_{j+1} \ \ldots \ p_k \\ & \vdots & \\ & \sim^2 & R_l \ldots R_2 \, p_j \, p_1 \ \ldots \ p_{j-1} \, x \, p_{j+1} \ \ldots \ p_k \end{array}$$

Therefore, Knuth relations exactly generate first row of $P(\pi)$. Also the element replaced by x from the first row comes at the end of R_2. The above sequence of operations can be repeated for each row to get the same tableaux, which completes the proof of Knuth's Theorem.

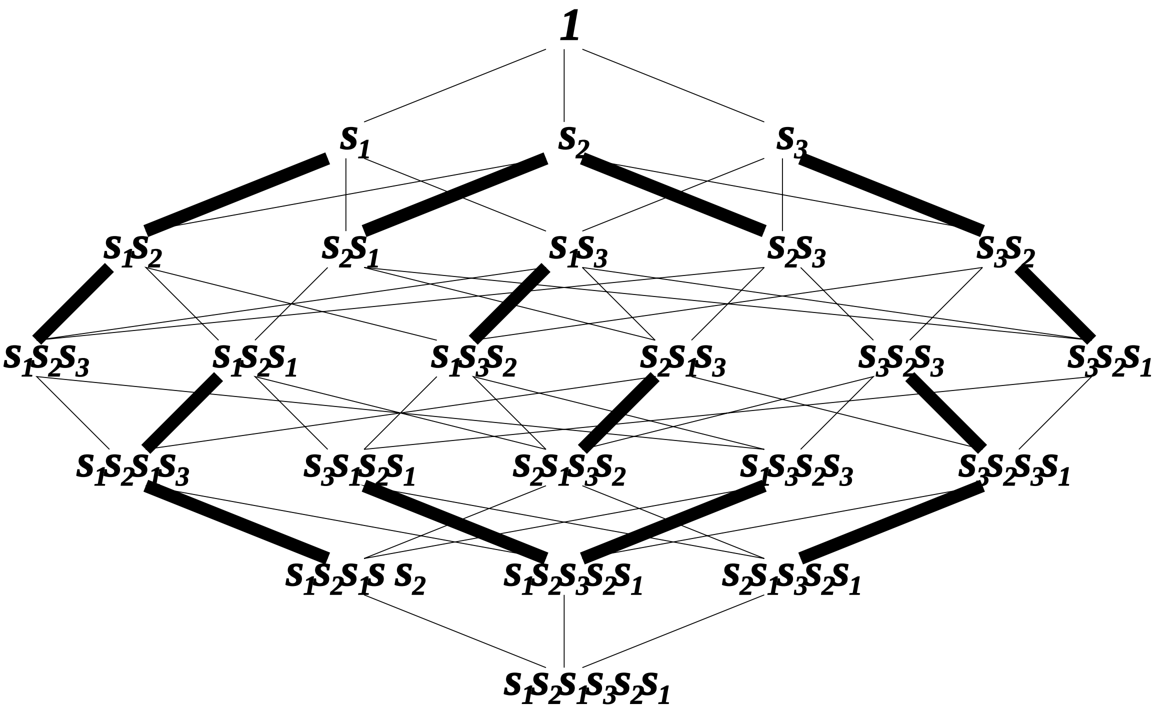
QED

3 Kazhdan Lusztig Cells

We aim to prove the following main result.

Theorem 3 *Let $\pi, \sigma \in \mathfrak{S}_n$. Then π and σ are in same left (resp. right) Kazhdan Lusztig cell iff $Q(\pi) = Q(\sigma)$ (resp. $P(\pi) = P(\sigma)$).*

For illustration, consider the following Bruhat-Chevalley graph on $\mathfrak{S}_4$, where transitive edges are omitted to make it more convenient to read. Elements of $\mathfrak{S}_n$ joined by thick edges are in same right cells.



3.1 Preliminary Notations

We first give a convenient way to write Knuth operations.
Let $D_{ij} := \{w \in \mathfrak{S}_n \ni ws_i < w,\ ws_j > w\}$, where $j = i \pm 1$. For any $y \in D_{ij}$, let y_0 be unique element of minimum length in right coset $y\langle s_i, s_j\rangle$. It is easy to check that we have only following two possibilities

$$y = y_0 s_i \textit{ or } y = y_0 s_j s_i$$

Define K_{ij} (Knuth operation) from D_{ij} to D_{ji} as:

$$K_{ij}(y) = \begin{cases} ys_j, & if \quad y = y_0 s_i \\ ys_i, & if \quad y = y_0 s_j s_i \end{cases}$$

It is easy to verify that these maps are bijections and nothing but elementary Knuth operations.

3.2 Preparatory Results

Lemma 1 *If $y, w \in \mathfrak{S}_n$ and $y \sim^1 w$ or $y \sim^2 w$, then $y \sim_R w$ (i.e, they are in same right cell).*

PROOF:
First assume that $y \sim^1 w$, which implies that there exist $y_{i+1} < y_i < y_{i+2}$, such that y and w have following forms:

$$y = y_1 \ \dots \ y_i \ y_{i+1} \ y_{i+2} \ \dots \ y_n$$

$$w = y_1 \ \ldots \ y_i \, y_{i+2} \, y_{i+1} \ \ldots \ y_n$$

which means $w = ys_{i+1} > y$ and hence $P_{y,w} = 1$. Therefore, $\mu(y,w) = 1$, which by definition means $y \prec w$. Also

$$s_{i+1} \in \mathscr{R}(w) \text{ and } s_{i+1} \notin \mathscr{R}(y)$$

$$s_i \in \text{ and } s_i \notin \mathscr{R}(w)$$

implying that y and w are in same right cell. The other case is similar.

QED

Proposition 1 $y \leq_L w \Rightarrow \mathscr{R}(w) \subset \mathscr{R}(y)$

Proof:
Case 1: $y \prec w$
That is, $\mu(y,w) \neq 0$ and $\mathscr{L}(y) \not\subset \mathscr{L}(w)$. Assuming the contrary, there exist $s \in S$ such that $ws < w$ but $ys > y$. In this case we have $P_{y,w} = P_{ys,w}$. Therefore, $\mu(y,w) \neq 0$ if and only if $w = ys > y$. But this directly implies that $\mathscr{L}(y) \subset \mathscr{L}(w)$, which is contradiction.

Case 2: $w \prec y$
This means that $\mu(w,y) \neq 0$ and $\mathscr{L}(y) \not\subset \mathscr{L}(y)$. Therefore, there exist some $s \in S$, such that $sw < w$ but $sy > y$. Following previous way of arguing, we have $y = sw$ and thus $\mathscr{R}(w) \subset \mathscr{R}(y)$.

QED

Proposition 2 *Let* $y \neq w \in D_{ij}$. *Then*

$$\mu(y,w) \neq 0 \Rightarrow \mu(K_{ij}(y), K_{ij}(w)) \neq 0$$

Corollary 3 *Let* $y, w \in D_{ij}$. *Then* $y \sim^L w$ *implies that* $K_{ij}(y) \sim^L K_{ij}(w)$.

3.3 Proof of Theorem 3

One implication is already proved in Lemma 1. That is,

$$y \sim^K w \Rightarrow y \sim^R w$$

It only remains to prove $y \sim^R w$ implies $P(y) = P(w)$. Or equivalently, $y \sim^L w$ implies $Q(y) = Q(w)$.

For each partition $\lambda \vdash n$, define standard tableaux P_λ by setting entries in i^{th} column to be

$$\sum_{j=1}^{i-1} \lambda_j + 1, \ \ldots \ \sum_{j=1}^{i} \lambda_j$$

where $\lambda = (\lambda_1, \dots, \lambda_k)$.

Now suppose λ_1 and λ_2 are shapes of $Q(y)$ and $Q(w)$ respectively. Define $\hat{y}$ and $\hat{w}$ by following inverse R-S correspondance

$$(P(\hat{y}), Q(\hat{y})) = (P_{\lambda_1}, Q(y))$$

$$(P(\hat{w}), Q(\hat{w})) = (P_{\lambda_2}, Q(w))$$

Since $Q(y) = Q(\hat{y})$, we have $y \sim^L \hat{y}$ and similarly $w \sim^L \hat{w}$. Transtivity of $\sim^L$ implies then that $\hat{y} \sim^L \hat{w}$. Again define y' and w'' by inverse R-S correspondance

$$(P(y'), Q(y')) = (P_{\lambda_1}, P_{\lambda_1})$$

$$(P(w''), Q(w'')) = (P_{\lambda_2}, P_{\lambda_2})$$

Since $P(y') = P(\hat{y})$, they are connected by Knuth relations and the same assertion is true for $\hat{w}$ and w''. Let

$$y' = K_{i_1,j_1} \circ \ldots \circ K_{i_r,j_r}(\hat{y})$$

$$w'' = K_{i'_1,j'_1} \circ \ldots \circ K_{i'_s,j'_s}(\hat{w})$$

Now define y'' and w' as

$$w' = K_{i_1,j_1} \circ \ldots \circ K_{i_r,j_r}(\hat{w})$$

$$y'' = K_{i'_1,j'_1} \circ \ldots \circ K_{i'_s,j'_s}(\hat{y})$$

To see that above definition makes sense, we know that $\hat{y} \sim^L \hat{w}$, therefore, $\mathscr{R}(\hat{y}) = \mathscr{R}(\hat{w})$, from Proposition 1. Therefore, we have

$$\hat{y} \in D_{i_r,j_r} \iff \hat{w} \in D_{i_r,j_r}$$

Thus we can define $K_{i_r,j_r}(\hat{w})$. From corollary to Proposition 2, we can proceed inductively to define w' (since, $K_{i_r,j_r}(\hat{y}) \sim^L K_{i_r,j_r}(\hat{w})$). Similar argument shows that definition of y'' also makes sense. This also proves the following statements

$$\begin{gathered} y' \sim^L w' \Rightarrow \mathscr{R}(y') = \mathscr{R}(w') \\ P(y') = Q(y') = P_{\lambda_1} \\ P(w') = P_{\lambda_2} \end{gathered} \tag{4}$$

$$\begin{gathered} y'' \sim^L w'' \Rightarrow \mathscr{R}(y'') = \mathscr{R}(w'') \\ P(y'') = P_{\lambda_1} \\ P(w'') = Q(w'') = P_{\lambda_2} \end{gathered} \tag{5}$$

From (4) and (5) respectively, we get the following

$$y' = \lambda_1 \; \lambda_1 - 1 \; \ldots \; 1 \; \lambda_1 + \lambda_2 \; \ldots \; \lambda_1 + 1 \; \ldots$$

$$w'' = \lambda'_1 \; \lambda'_1 - 1 \; \ldots \; 1 \; \lambda'_1 + \lambda'_2 \; \ldots \; \lambda'_1 + 1 \; \ldots$$

where $\lambda_2 = (\lambda'_1, \dots, \lambda'_{k'})$. Now $\mathscr{R}(y') = \mathscr{R}(w')$ implies that first λ_1 entries of w' are decreasing, and next λ_2 entries of w' are decreasing and so on. The similar statement holds for y'', since $\mathscr{R}(w'') = \mathscr{R}(y'')$. Together they imply that $\lambda_1 = \lambda_2$ and hence $y' = w'$. This coupled with definitions of y' and w' give us

$$\hat{y} = \hat{w} \Rightarrow Q(\hat{y}) = Q(\hat{w})$$

and since $Q(\hat{y}) = Q(y)$ and $Q(\hat{w}) = Q(w)$, we get the other implication of Theorem 3