Complex Semi-simple Lie Algebras

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Abstract

These notes are towards structure theory of complex semisimple Lie algebras. We prove root space decomposition of semisimple Lie algebras and associate a root system (which determines its structure). We also prove existence of base of root system and its association with Weyl chambers. The classfication theorem for root systems will be proved in notes on Root Systems (under preparation), and inverse correspondance will be given (under some other title). All algebras and vector spaces considered in these notes are over field of complex numbers and are finite dimensional.

1 Lie Algebras: Introduction

The aim of this section is to give basic definitions, introduce solvable and nilpotent Lie algebras and prove Engel's Theorem and Lie's Theorem.

1.1 Some Definitions

This part contains several definitions and no proofs (which are fairly trivial). Just go through them.

Let $\mathfrak{g}$ be a $\mathbb{C}$-algebra, where product is written as bracket $((X,Y) \longmapsto [X,Y])$. We say $\mathfrak{g}$ is *Lie algebra* if following relations hold:

$$[X,X] = 0;\ \forall X \in \mathfrak{g} \tag{1}$$

$$[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0,\ \forall X,Y,Z \in \mathfrak{g} \tag{2}$$

A subspace $\mathfrak{g}'$ of $\mathfrak{g}$ is called *Lie subalgebra*, if it is closed under bracket (i.e Lie algebra under multiplication induced from $\mathfrak{g}$). A subalgebra $\mathfrak{a}$ is called *ideal* of $\mathfrak{g}$ if for every $X \in \mathfrak{g}$ and $Y \in \mathfrak{a}$, $[X,Y] \in \mathfrak{a}$. Note that (1) implies that every ideal is always two-sided, therefore, we need not define right and left ideals separately. Note that if $\mathfrak{a}$ is an ideal, then $\mathfrak{g}/\mathfrak{a}$ has canonical structure of Lie algebra (called *quotient algebra*)

Center of Lie algebra is defined to be $Z(\mathfrak{g}) := \{X \in \mathfrak{g} : [X,Y] = 0,\ \forall Y \in \mathfrak{g}\}$. $\mathfrak{g}$ is called *abelian* if $Z(\mathfrak{g}) = \mathfrak{g}$. We alse need notion of *derived subalgebra*, denoted by $[\mathfrak{g},\mathfrak{g}]$, which is subalgebra generated by $\{[X,Y] : X,Y \in \mathfrak{g}\}$. Note that derived algebra is always an ideal and the quotient $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ is an abelian Lie algebra. Moreover, if $\mathfrak{a}$ is an ideal such that $\mathfrak{g}/\mathfrak{a}$ is abelian, then $[\mathfrak{g},\mathfrak{g}] \subseteq \mathfrak{a}$.

We leave definitions of Lie algebra morphisms, isomorphisms, kernel etc. since these are fairly standard. Let $\mathfrak{M}_n$ denote all $n \times n$ matrices with complex entries. We can define bracket operation on it as follows: $[X,Y] = XY - YX$, where $X,Y \in \mathfrak{M}_n$. This defines a structure of Lie algebra on $\mathfrak{M}_n$, which will be denoted by $\mathfrak{gl}_n(\mathbb{C})$. Moreover, for any complex vector space V, we can define structure of Lie algebra on endomorphism ring $End(V)$, and denote it by $\mathfrak{gl}(V)$.

A *representation* of Lie algebra $\mathfrak{g}$ is a Lie algebra homomorphism $\varphi : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ where V is some finite dimensional complex vector space. Let $ad : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$ be map defined by $ad(X)(Y) = [X,Y]$. This representation is called *adjoint representation*. Note that $Ker(ad) = Z(\mathfrak{g})$. Therefore, if center of $\mathfrak{g}$ is trivial, it can be viewed as subalgebra of $\mathfrak{gl}(\mathfrak{g})$. A Lie subalgebra of $\mathfrak{gl}(V)$ is called *linear Lie algebra*. For example, the set of all $n \times n$ upper triangular matrices (resp. strict upper triangular matrices, diagonal matrices) is linear Lie algebra denoted by $\mathfrak{t}(n,\mathbb{C})$ (resp. $\mathfrak{n}(n,\mathbb{C})$, $\mathfrak{d}(n,\mathbb{C})$).

1.2 Solvable and Nilpotent Lie algerbas

Let L be a Lie algebra. We define two series of ideals in L as follows:

Definition 1 Let $L^{(0)} = L$ and $L^{(i)} = [L^{(i-1)}, L^{(i-1)}]$, for each $i \geq 1$. We get the following series, called *derived series* of L.

$$L = L^{(0)} \subseteq L^{(1)} \subseteq \dots$$

We say L is *solvable* if $L^{(n)} = 0$, for some n.

Definition 2 Let $L^0 = L$ and $L^i = [L, L^{i-1}]$, for each $i \geq 1$. We get the following series, called *lower central series* of L.

$$L = L^0 \subseteq L^1 \subseteq L^2 \subseteq \dots$$

We say L is *nilpotent* if $L^n = 0$ for some n.

One immediate consequence worth noting is that every subalgebra and quotient algebra of solvable (resp. nilpotent) Lie algebra is itself solvable (resp. nilpotent). In solvable case, converse also holds, that is, if I is an ideal of L, such that both I and L/I are solvable, then so is L. In case of nilpotent, we have: if $L/Z(L)$ nilpotent, then so is L and $Z(L) \neq 0$ (provided $L \neq 0$). An immediate consequence of all this is that in any Lie algebra L, if I, J are solvable ideals then so is $I + J$. Hence, there is unique maximal solvable ideal (called *radical of* L), denoted by $Rad(L)$. We say L is *semisimple* if $Rad(L) = 0$. Note that it is equivalent to saying L has no abelian ideals.

1.3 Engel's Theorem

Let L be a nilpotent Lie algebra. Take $n > 0$ such that $L^n = 0$. Then it is clear that for any $X \in L$, we have $\underbrace{[X, [X, \dots, [X, Y] \dots]}_{n\ times} = 0$, for any $Y \in L$. This is equivalent to say that $ad(X)^n = 0$ or $ad(X)$ is nilpotent endomorphism of L. Hence we deduce that if L is nipotent, then every element of L is *ad-nilpotent* (we say $X \in L$ is *ad-nilpotent* if $ad(X)$ is nilpotent endomorphism of L). This section will primarily concentrate on proof of converse of this statement.

Theorem 1 *[Engel] If all elements of L are ad-nilpotent, then L is nilpotent Lie algebra.*

Lemma 1 *If $x \in \mathfrak{gl}(V)$ is nilpotent endomorphism, then $ad(x)$ is also nilpotent.*

Proof:
Let λ_x and ρ_x denote left and right multiplication by x respectively, which are elements of endomorphisms of $End(V)$. Now $\lambda_x(\rho_x(y)) = xyx = \rho_x(\lambda_x(y))$ and hence they commute.

Moreover $ad(x)(y) = xy - yx = (\lambda_x - \rho_x)(y)$. Since λ_x and ρ_x are nilpotent commuting endomorphisms, their difference is also nilpotent. That is, if n is such that $x^n = 0$, then

$$ad(x)^N = (\lambda_x - \rho_x)^N = \sum_{i=0} i = N \begin{pmatrix} N \\ i \end{pmatrix} \lambda_x^{N-i} \rho_x^i = 0$$

where $N \geq 2n + 1$.

QED

Observe that converse of this lemma is not true, since identity is ad-nilpotent but not nilpotent.

Theorem 2 *Let L be subalgebra of $\mathfrak{gl}(V)$. If $V \neq (0)$ and L consists of nilpotent endomorphisms, then there is some non-zero $v \in V$ such that $L.v = 0$.*

PROOF:
We prove the theorem by induction on dimension of L. In case L is one dimensional, say generated by x, then it always has one eigenvector corresponding to eigenvalue 0, which proves the theorem.

Now let $K \neq L$ be any subalgebra of L. According to previous lemma, every element of K is ad-nilpotent on L (K acts on L via ad- map) and hence on L/K (vector space). Since $dim(K) < dim(L)$, we can use induction hypothesis to assert that there is some element $x + K \in L/K$ such that $ad(K)(x + K) = 0$. That is, there is some non-zero element $x \in L \setminus K$, such that $[K, x] \subseteq K$. We deduce that $K \subsetneq N_L(K)$, where $N_L(K)$ is normalizer of K, which consists of elements $y \in L$ such that $[K, y] \subseteq K$.

Let K be maximal proper subalgebra of L. The previous argument gives that $N_L(K) = L$, i.e, K is an ideal of L. We claim that it is of codimension 1. Assume the contrary, that $dim(L/K) > 1$. Then inverse image of any one dimensional subalgebra of L/K is subalgebra of L properly containing K, which contradicts the maximality of K. Hence, we can write $L = K + \mathbb{C}z$, where $z \in L \setminus K$.

Now let $W = \{v \in V : K.v = 0\}$, which is non-zero by induction hypothesis. Moreover, W is stable under L, since for any $w \in W$, $y \in K$ and $x \in L$, we have (as $[x, y] \in K$):

$$y(x.w) = x(y.w) + [y, x]w = 0$$

Hence, we can find a non-zero eigenvector of z in W corresponding to its eigenvalue 0, say w. Then $L.w = 0$ and hence theorem is proved.

QED

Now we return to the proof of Theorem 1.

PROOF:

We are given L, Lie algebra whose all elements are ad-nilpotent. That is, $ad(L) \subseteq \mathfrak{gl}(L)$ consists of nilpotent matrices. Applying previous theorem, we can conclude that: there is some element $x \in L$ such that $[L, x] = 0$, which means $Z(L) \neq 0$. Now $L/Z(L)$ is Lie algebra of smaller dimension which again consists of ad-nilpotent elements and hence is nilpotent by induction hypothesis. By remarks at end of previous section this implies L itself is nilpotent.

QED

Theorem 2 has interesting consequences. Assuming its notations, we find a non-zero element of V, say v_n, such that $L.v_n = 0$. Let $V_{n-1} = V/\mathbb{C}v_n$. Again we can find some non-zero element of V_{n-1}, say v_{n-1} with same property. Proceeding like this we can find a chain of subspaces of V

$$0 = V_0 \subset V_1 \subset \ldots \subset V_n = V$$

such that $L.V_i \subset V_{i-1}$, i.e, elements of L look like strict upper triangular matrices, with respect to basis consisting of $v_1, \ldots, v_n$. Another important consequence is noted as:

Corollary 1 *Let L be nilpotent Lie algebra and K be an ideal of L. Then if $K \neq 0$, then $K \cap Z(L) \neq 0$.*

PROOF:

L acts on K via adjoint representation. Using Theorem 2, we get a non-zero element of K, killed by all of L, i.e, $x \in K$, such that $[L, x] = 0$, which implies x is in $Z(L)$.

QED

1.4 Lie's Theorem

We have proved in previous section that nilpotent linear Lie algebras are subalgebras of $\mathfrak{n}(n, \mathbb{C})$. Similar result is true for solvable case too, i.e, with respect to suitable basis, elements of solvable linear Lie algebra are all upper triangulars. This is Lie's Theorem, which we will prove in this section.

Theorem 3 *Let L be solvable subalgebra of $\mathfrak{gl}(V)$. If $V \neq 0$, then V contains a common eigenvector for all elements of L.*

PROOF:

The proof is organized in various steps imitating proof of Theorem 1. First we observe that $L/[L, L]$ is abelian and therefore, every subspace is automatically an ideal. Now take inverse image of a subspace of $L/[L, L]$ of codimension on (under canonical map $L \to L/[L, L]$). This is an ideal of L of codimension one and containing $[L, L]$, call it K.

Write $L = K + \mathbb{C}z$ and use induction to find common eigenvector of K. Thus there exists $\lambda \in K^*$, linear function, such that

$$W = \{v \in V : x.v = \lambda(x)v,\ \forall x \in K\}$$

is non trivial. We claim that L stablizes W. Let $w \in W,\ x \in L$. For any $y \in K$, we have

$$yx(w) = xy(w) - [x, y](w) = \lambda(y)x(w) - \lambda([x, y])w$$

Thus we have to prove that $\lambda([x, y]) = 0$, for every $x \in L$ and $y \in K$. For this fix $x \in L$ and $w \in W$. Let n be smallest positive integer such that $w, xw, x^2w, \ldots, x^nw$ are linearly dependent. Denote by W_i, the subspace of W generated by $\{w, xw, \ldots, x^{i-1}w\}$. x maps each W_i to W_{i+1} $(1 \leq i \leq n-1)$ and stablizes W_n. We first prove the following congruence, for every $y \in K$

$$yx^iw \equiv \lambda(y)x^iw\ (mod\ W_i)$$

Writ $yx^iw = yxx^{i-1}w = xyx^{i-1}w - [x, y]x^{i-1}w$. Now apply induction to write yx^{i-1} as $\lambda(y)x^{i-1}w + w'$, where $w' \in W_{i-1}$. Since each element of K stablizes each W_i and x maps W_{i-1} to W_i, we get the desired congruence relation. This proves that relative to the basis $\{w, xw, \ldots, x^{n-1}w\}$, each $y \in K$ is represented by upper triangular matrix with diagonal entries $\lambda(y)$. Therefore, $Tr_{W_n}(y) = n\lambda(y)$ holds for each $y \in K$. In particular for any $y \in K$ $Tr_{W_n}([x, y]) = n\lambda([x, y])$. But since both x and y stablize W_n, their commutator has zero trace, which proves that $\lambda([x, y]) = 0$.

Now we can finish the proof by finding some eigenvector for z in W, which will be common eigenvector for whole of L.

QED

The next corollary is obvious from Theorem 3, by using induction on dimension of V.

Corollary 2 *[Lie's Theorem] Let L be solvable Lie algebra of $\mathfrak{gl}(V)$. Then L stablized some flag in V. That is, there is a flag $0 = V_0 \subset V_1 \subset \ldots \subset V_n = V$, such that $L.V_i \subseteq V_i$.*

2 Semisimple Lie Algebras-1

In this section, we clear the ground work in order to study structure of complex semisimple Lie algebras. The basic idea of all this is to first prove that every semisimple Lie algebra is linear. This is done by observing that it has zero center (since it contains no nontrivial abelian ideals) and hence adjoint representation is faithful. Therefore, if we can find an abelian subalgebra, whose every element is ad-semisimple, then adjoint representation will break the Lie algebra into so-called root spaces (eigenspaces for this subalgebra acting on original Lie algebra). Moreover, if we require this subalgebra to be maximal possible, then root spaces are each one dimensional and structure of this subalgebra completely determines structure of original algebra (via root space decomposition).

2.1 Jordan Decomposition

Recall the following result regarding Jordan decomposition. Any matrix M over a perfect field K, can be written uniquely as sum of nilpotent matrix (N) and semisimple[1] matrix (S), which commute ($SN = NS$). Moreover, both N and S are polynomials in M, which are called nilpotent and semisimple parts of M respectively.

Lemma 2 *If $x \in End(V)$ and $x = x_s + x_n$ is its Jordan decomposition, then $ad(x) = ad(x_s) + ad(x_n)$ is Jordan decomposition of $ad(x)$ in $End(End(V))$.*

Proof:
First we recall from previous section, that if n is nilpotent endomorphism of V, then $ad(n)$ is nilpotent endomorphism of $End(V)$. We claim that same result holds for semisimple endomorphisms. Choose a basis of V (say $\{v_1, \dots, v_n\}$) with respect to which x_s can be written as diagonal matrix (say $diag(a_1, \dots, a_n)$). Take canonical basis of $End(V)$, $\{e_{ij}\}$ with respect to this chosen basis of V, that is, $e_{ij}(v_k) = \delta_{jk} v_i$. Then it is trivial to verify that $ad(x_s)(e_{ij}) = (a_i - a_j)e_{ij}$ and hence $ad(x_s)$ is semisimple.

Now $[ad(x_s), ad(x_n)] = ad([x_s, x_n]) = 0$ and $ad(x) = ad(x_s) + ad(x_n)$. From uniqueness of Jordan decomposition, we get that this is the Jordan decomposition of $ad(x)$.

QED

Lemma 3 *If $\mathfrak{A}$ is finite dimensional $\mathbb{C}$-algebra, then $Der(\mathfrak{A})$ [2] contains semisimple and nilpotent parts of all of its elements.*

[1] Note that in case of algebraically closed field, semisimple is same as diagonizable

[2] A derivation of Lie algebra L is an endomorphism δ, such that $\delta([X, Y]) = [\delta(X), Y] + [X, \delta(Y)]$. Note that derivations of L form subalgebra of $\mathfrak{gl}(L)$

2.2 Killing Form

Definition 3 Let $\pi : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ be any finite dimensional representation of $\mathfrak{g}$. Then we associate a symmetric invariant [3] bilinear form on $\mathfrak{g}$ as follows:

$$(X, Y) \longmapsto Tr(\pi(X) \circ \pi(Y))$$

In particular when π is adjoint representation, we call it *Killing form.*

Lemma 4 *Let A, B be two subspaces of $\mathfrak{gl}(V)$. Set $M = \{x \in \mathfrak{gl}(V) : [x, B] \subseteq A\}$. Suppose $x \in M$ satisfies $Tr(xy) = 0$ for every $y \in M$. Then x is nilpotent.*

Proof:
Let $x = s + n$ be Jordan decomposition. It suffices to prove that $s = 0$. So fix a basis $\{v_1, \ldots, v_n\}$ of V, with respect to which s has form $diag(a_1, \ldots, a_n)$. Set E to be $\mathbb{Q}$-span of $\{a_1, \ldots, a_n\}$. It suffices to prove that $E = 0$ or equivalently $E^* = 0$. That is, every linear functin $f : E \to \mathbb{Q}$ is zero.

Let $y \in \mathfrak{gl}(V)$ be endomorphism, whose matrix with respect to chosen basis is $diag(f(a_1), \ldots, f(a_n))$. If we can prove that $y(B) \subseteq A$ (i.e, $y \in M$), then we will get that $Tr(xy) = \sum a_i f(a_i) = 0$. Applying f yields, $\sum f(a_i)^2 = 0$ and hence all of $f(a_i) = 0$, which will prove the lemma. Now we have

$$ad(s)(e_{ij}) = (a_i - a_j)e_{ij},\ ad(y)(e_{ij}) = (f(a_i) - f(a_j))e_{ij}$$

where $\{e_{ij}\}$ is canonical basis of $\mathfrak{gl}(V)$ with respect to chosen basis of V. Let $r(T) \in (C)[T]$, be polynomial without constant term, satisfying $r(a_i - a_j) = f(a_i) - f(a_j)$. Then we have $ad(y) = r(ad(s))$. Since $ad(s)$ is semisimple part of $ad(x)$, it can be written as polynomial in $ad(x)$ and hence $ad(y)$ can be written as polynomial in $ad(x)$. By hypothesis, $ad(x)$ maps B into A and hence so is true for $ad(y)$. Thus we have proved $y \in M$ and this completes proof of lemma.

QED

Theorem 4 *Let L be subalgebra of $\mathfrak{gl}(V)$. Suppose $Tr(xy) = 0$ for every $x \in [L, L]$ and $y \in L$. Then L is solvable.*

Proof:
It suffices to prove that $[L, L]$ is nilpotent, or just that all elements of $[L, L]$ are nilpotent endomorphisms (by Theorem 1). For this apply previous lemma for $A = [L, L]$, $B = L$, and thus $M = \{x \in \mathfrak{gl}(V) : [x, L] \subset [L, L]\}$. Therefore, in order to conclude that x is nilpotent from lemma, we need to show that for each $x \in [L, L]$, $y \in M$, we have $Tr(xy) = 0$.

[3] A bilinear form $(.,.)$ on Lie algebra $\mathfrak{g}$ is called invariant, if $([X, Y], Z) = (X, [Y, Z])$

If $[x, y]$ is some generator of $[L, L]$ and $z \in M$, then we have

$$Tr([x, y]z) = Tr(x[y, z]) = Tr([y, z]x) = 0$$

and hence the theorem.

QED

Theorem 5 *A Lie algebra L is semisimple if and only if Killing form (denoted by $\kappa(.,.)$) is non-degenerate.*

Proof:
First assume L is semisimple, i.e, $Rad(L) = 0$. Let $S = \{x \in L : \kappa(x, y) = 0,\ \forall y \in L\}$. Then for any $x \in S$ and $y \in L$, we have $Tr(ad(x)ad(y)) = 0$ and hence $ad_L(S)$ is solvable Lie algebra (by Theorem 4). Since ad is faithful, we get S is solvable. Therefore, $S \subset Rad(L) = 0$.

Conversely, assume $S = 0$. Let I be any abelian ideal of L. Suppose $x \in I$, $y \in L$. Then $ad(x)ad(y)$ maps L into I and since I is abelian, $(ad(x)ad(y))^2 = 0$ and hence has zero trace. This proves that $I \subset S = 0$. Therefore, L has no nontrivial abelian ideals and hence it is semisimple.

QED

Theorem 6 *Let L be semisimple Lie algebra. Then there exist ideals $L_1, \ldots, L_t$ of L, each of which is simple (as Lie algebra) and $L = L_1 \oplus \ldots \oplus L_t$. Moreover, every simple ideal of L is one of $L_i's$.*

Proof:
Let I be any ideal of L. Then $I^\perp := \{x \in L : \kappa(x, y) = 0,\ \forall y \in I\}$, is also an ideal. Since $\kappa|_{I \cap I^\perp}$ is zero, we have $I \cap I^\perp = (0)$. This proves that $L = I \oplus I^\perp$.

Now we proceed by induction on dimension of L. If L has no nontrivial ideals, then it is simple by definition. Assume L_1 is minimal non trivial ideal of L, then L_1 is simple and by what we have seen $L = L_1 \oplus L_1^\perp$. By induction hypothesis, $L_1^\perp$ breaks as sum of simple pieces, which proves the first part of theorem.

Assume I is any simple ideal of L. Then $[I, L] \subseteq I$ and is non-zero, since center is trivial. Since I is simple, we have $[I, L] = I$. But $[I, L] = [I, L_1] \oplus \ldots \oplus [I, L_t]$, therefore, all but one summand must be zero. This implies $I \subset L_i$ for certain i, which by simplicity of L_i gives $I = L_i$. This completes the proof.

QED

Note that $[L, L]$ is an ideal of L, and from proof of above theorem, it must be sum of certain simple ideals. If it is proper, then there is some simple ideal not contained in $[L, L]$, which should be isomorphic to $L/[L, L]$ and hence abelian. This contradicts definition of semisimplicity. Moreover, $[L, L]$ can't be zero, since center of L is trivial. Therefore, we have $L = [L, L]$. This also proves that if $\varphi : L \to \mathfrak{gl}(V)$ is a representation, then every element of $\varphi(L)$ has zero trace, i.e, $\varphi(L) \subset \mathfrak{sl}(V)$.

Theorem 7 *If L is semisimple, then all its derivations are inner, i.e, $Der(L) = ad(L)$.*

PROOF:
Since L is semisimple, $L \to ad(L)$ is isomorphism of Lie algebras. Hence, $ad(L)$ itself has non-degenerate Killing form. Let I be orthogonal to $ad(L)$ in $Der(L)$ with respect to Killing form. Since $ad(L)$ and I are both ideals in $Der(L)$ and their intersection is trivial (by non-degeneracy of Killing form on $ad(L)$), we have $[I, ad(L)] = 0$. This proves that for any $\delta \in I$, we have $[\delta, ad(x)] = 0 \Rightarrow ad(\delta x) = 0$. Since ad is faithful, this implies $\delta(x) = 0$ for every $x \in L$, which implies $\delta = 0$. Thus, $I = 0$ and $Der(L) = ad(L)$.

QED

2.3 Weyl's Theorem of complete reducibility

First we introduce *Casimir operator* for any faithful representation of semisimple Lie algebra. Let $\phi : L \to \mathfrak{gl}(V)$ be a faithful representation. Let β be any non-degenerate, invariant, symmetric bilinear form on L. For any basis $\{x_1, \ldots, x_n\}$ of L, we can select a dual basis (with respect to β), say $\{y_1, \ldots, y_n\}$. Define an element of $\mathfrak{gl}(V)$, $c_\phi(\beta)$ as

$$c_\phi(\beta) := \sum_{i=1}^{n} \phi(x_i)\phi(y_i)$$

For any faithful representation of L, we have a canonical bilinear form defined as $\beta(x, y) = Tr(\phi(x)\phi(y))$. In this case $c_\phi(\beta)$ or simply c_ϕ is called *Casimir element* of ϕ. We note that trace of Casimir element is same as dimension of L.

Next we relate for any $x \in L$, $[x, x_i]$ and $[x, y_i]$. Assume we write them as:

$$[x, x_i] = \sum_{j=1}^{n} a_{ij} x_j, \; [x, y_i] = \sum_{j=1}^{n} b_{ij} y_j$$

Then using invariant of bilinear form β, as defined we get:

$$a_{ik} = \sum_j a_{ij} \beta(x_j, y_k) = \beta([x, x_i], y_k) = -\beta(x_i, [x, y_k]) = -b_{ki}$$

Therefore, for any representation ϕ (as written above), and for any $x \in L$, we can compute

$$[x, c_\phi] = \sum_i [\phi(x), \phi(x_i)]\phi(y_i) + \sum_i \phi(x_i)[\phi(x), \phi(y_i)]$$

which is same as $\sum_{i,j} a_{ij}\phi(x_j)\phi(y_i) + \sum_{i,j} b_{ij}\phi(x_i)\phi(y_j)$ and from the relation proved above $a_{ij} = -b_{ji}$, it is zero. This proves that c_ϕ commutes with elements of $\phi(L)$.

Theorem 8 *[Weyl] Let $\phi : L \to \mathfrak{gl}(V)$ be any finite dimensional representation of semisimple Lie algebra L. Then ϕ is completely reducible.*[4]

PROOF:
Let W be sub-L-module of V of codimension one. Since L acts trivially on one dimensional modules, we have following exact sequence of L-modules

$$0 \longrightarrow W \longrightarrow V \longrightarrow \mathbb{C} \longrightarrow 0$$

In subcase, when W is irreducible, $c = c_\phi$ be Casimir element, is L-module endomorphism of V (since it commutes with $\phi(L)$. Therefore, $c(W) \subset W$ and $Ker(c)$ is L-submodule of V. Because L acts trivially on V/W, so does c (as it is linear combination of product of elements $\phi(x)$). By Schur's lemma, c must act as scalar on W and this scalar can't be zero (since $tr(c) = dim(L)$). Hence, $Ker(c)$ is one dimensional submodule of V, which intersects W trivially, which gives desired decomposition.

We are still in case where W is of codimension one. Assume it is not irreducible. Then we proceed by induction on dimension of W. If W' is proper non-zero submodule of W, we get another exact sequence of L-modules

$$0 \longrightarrow W/W' \longrightarrow V/W' \longrightarrow \mathbb{C} \longrightarrow 0$$

By induction, we can write $V/W' = W/W' \oplus \tilde{W}/W'$. Hence, we have another exact sequence

$$0 \longrightarrow W' \longrightarrow \tilde{W} \longrightarrow \mathfrak{C} \longrightarrow 0$$

We can apply induction hypothesis again, since W' is of codimension one in $\tilde{W}$, with dimension strictly less than that of W. This provides us with $\tilde{W} = W' \oplus X$. Therefore we get required decomposition $V = W \oplus X$.

Now we come to general case. Consider the following (where W is arbitrary L-submodule of V)

$$0 \longrightarrow W \longrightarrow V \longrightarrow V/W \longrightarrow 0$$

[4] meaning, it breaks as direct sum of irreducible representations, or equivalently, for any submodule V' of V, there is another submodule V'', such that $V = V' \oplus V''$

Consider $Hom(V,W)$ be space of linear maps, which has structure of L-module[5] Consider the subspace $\mathscr{V}$ of $Hom(V,W)$, consisting of those linear maps whose restriction to W is scalar, and subspace $\mathscr{W}$ of $Hom(V,W)$, consisting of linear maps whose restriction to W is zero. For any $f \in Hom(V,W)$ such that $f|_W = a.1_W$, we have (for $x \in L$)

$$(x.f)(w) = x.(f(w)) - f(x.w) = 0$$

Hence $\mathscr{V}$ is actually L-submodule and L maps $\mathscr{V}$ into $\mathscr{W}$. Moreover, $\mathscr{W}$ is of codimension one in $\mathscr{V}$, which is again the situation we have already dealt. Therefore we can find one dimensional L-submodule of $\mathscr{V}$, which is complementary to $\mathscr{W}$. Let $f : V \to W$ be any generator of this one-dimensional L-module. We can assume that $f|_W = 1_W$. Since L annihilates f (L acts trivially on one dimensional spaces), f is L-module map, which maps V into W and is identity on W. Hence $Ker(f)$ is L-submodule, complementary to W.

QED

For semisimple Lie algebras, we can introduce Jordan decomposition, by identifying L with $Der(L)$. Let $ad(x) = ad(x_s) + ad(x_n)$ be Jordan Decomposition in $Der(L)$. We call x_s and x_n, the semisimple and nilpotent parts of x respectively. (We have already proved that $Der(L)$ contains semisimple and nilpotent parts of all of its elements). This might cause problem in case L happens to be linear semisimple Lie algebra and we will have to check whether the usual Jordan decomposition in L is same as the decomposition just introduced. The next theorem is step towards this.

Theorem 9 *Let $L \subset \mathfrak{gl}(V)$ be linear semisimple Lie algebra. Then L contains semisimple and nilpotent parts in $\mathfrak{gl}(V)$ of all its elements.*

PROOF:
Let $x \in L$ be any element, with Jordan decomposition $x = s + n$ in $\mathfrak{gl}(V)$. We will prove that $s, n \in L$. Set $N = N_{\mathfrak{gl}(V)}(L) = \{y \in \mathfrak{gl}(V) : ad(y)(L) \subset L\}$. We have $x \in N$ and since $ad(s)$ and $ad(n)$ are polynomials in $ad(x)$, $s, n \in N$. Now for any L-submodule W of V, define $L_W := \{y \in \mathfrak{gl}(V) : yW \subset W,\ Tr(y|_W) = 0\}$. Define L' to be intersection of N with all L_W. Clearly, $L \subset L'$ and $s, n \in L'$. It suffices to prove that $L = L'$.

Since L' is finite dimensional L-module, it can be written as $L' = L + M$. Since, $[L, L'] \subset L$, L acts trivially on M. If $y \in M$, then for any irreducible L-submodule W of V, $[L, y] = 0$ gives us $y|_W$ is scalar and since its trace is zero, it must be zero. But V itself is direct sum of irreducible submodules. Hence $y = 0$, which in turn implies $L = L'$.

QED

[5]If $M_1, \dots, M_n, N$ are L-modules, the space of n-linear maps $Hom(M_1 \times \dots \times M_n; N)$ has L-module structure defined by $(x.f)(m_1, \dots, m_n) = x.(f(m_1, \dots, m_n)) - \sum_i f(m_1, \dots, m_{i-1}, x.m_i, m_{i+1}, \dots m_n)$, for any $x \in L$, $m_i \in M_i$, $\forall i$, $1 \le i \le n$

Corollary 3 *If L is semisimple Lie algebra and $\phi : L \to \mathfrak{gl}(V)$ is finite dimensional representation of L, then $x = s + n$, abstract Jordan decomposition in L implies that $\phi(x) = \phi(s) + \phi(n)$ is usual Jordan decomposition in $\mathfrak{gl}(V)$.*

PROOF:
Since L is spanned by eigenvectors of $ad_L(s)$, we have $\phi(L)$ is spanned by eigenvectors of $ad_{\phi(L)}\phi(s)$ and hence $ad_{\phi(L)}\phi(s)$ is semisimple. Similarly, $ad_{\phi(L)}\phi(n)$ is nilpotent. Therefore, $\phi(x) = \phi(s)+\phi(n)$ is abstract Jordan decomposition and hence by theorem the usual Jordan decomposition (since $\phi(L)$ is semisimple Lie algebra).

QED

3 Semisimple Lie Algebras-2

3.1 $\mathfrak{sl}_2(\mathbb{C})$

Let L denote $\mathfrak{sl}_2(\mathbb{C})$ (which is 2×2, traceless matrices over field of complex numbers), whose standard basis is

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We have the following relations

$$[h, x] = 2x,\ [h, y] = -2y,\ [x, y] = h$$

Let V be L-module (finite dimensional always). Since h is semisimple, we have already seen that h will act diagonally on V. Therefore, we get a decomposition by setting $V_\lambda = \{v \in V : h.v = \lambda v\}$, where $\lambda \in \mathbb{C}$. Whenever $V_\lambda \neq 0$, we say V_λ is a *weight space* and λ is a weight of h on V.

Now since $[h, x] = 2x$ and $[h, y] = -2y$, we have for any non zero $v \in V_\lambda$,

$$h(x.v) = x(h.v) + [h, x]v = (\lambda + 2)xv$$

$$h(y.v) = y(h.v) + [h, y]v = (\lambda - 2)xv$$

and hence $x.v \in V_{\lambda+2}$ and $y.v \in V_{\lambda-2}$. Since V is finite dimensional and $V = \oplus_\lambda V_\lambda$ is direct, we have that x and y are nilpotent endomorphisms on V. Let V_λ be weight space such that $V_{\lambda+2} = 0$ (or, $x.V_\lambda = 0$). Such a λ is called *maximal weight* and any non zero element of V_λ is called *maximal vector* of weight λ.

Assume now that V is irreducible L-module. Choose a maximal vector $v_0 \in V_\lambda$. Set

$$v_{-1} = 0,\ v_i = \left(\frac{1}{i!}\right) y^i.v_0$$

Then we have following relations which can be verified trivially

$$h.v_i = (\lambda - 2i)v_i \tag{3}$$

$$y.v_i = (i + 1)v_{i+1} \tag{4}$$

$$x.v_i = (\lambda - i + 1)v_{i-1} \tag{5}$$

Now we note some important observations, collected as Theorem 10, below. First observe that (3) implies that each v_i are in different weight spaces and hence are linearly independent. Moreover, all of these formulae, collectively imply that span of v_i's is stable under L action and hence form submodule. But since V is irreducible, it must be whole of V.

Since V is finite dimensional, choose m such that $v_m \neq 0$ but $v_{m+1} = 0$. Hence, we have a basis for V as $\{v_0, \ldots, v_m\}$. Now (5) applied for $i = m+1$ gives that $0 = (\lambda - m)v_m$. Hence weight of maximal vector is m, a positive integer, which is equal to $dim(V) - 1$. Moreover, any weight space is 1-dimensional. To summarize:

Theorem 10 *Let V be irreducible L-module.*

(a) Relative to h, V is direct sum of weight spaces V_μ, where $\mu = m, m-1, \ldots, -(m-2), -m$. Moreover, $m = dim(V) - 1$ and $dim(V_\mu) = 1$, for each μ.

(b) V has (upto scalar multiplication) a unique maximal vector, whose weight (called highest weight of V) is m.

(c) The action of L on V is explicitly given by formulae (3), (4), (5). Hence, there exists unique irreducible L-module (upto isomorphism) of possible dimension $m+1$, $m \geq 0$.

Corollary 4 *Let V be any finite dimensional L-module. Then eigenvalues of h on V are all integers, and each occurs with its negative (an equal number of times). Moreover, in any decomposition of V into direct sum of irreducible submodules, the number of summands is exactly equal to $dim(V_0) + dim(V_1)$.*

3.2 Cartan Subalgebras

From this section onwards, L is arbitrary semisimple Lie algebra. We will use adjoint representation of L to study its structure.

If L has only ad-nilpotent elements, then it is nilpotent itself. Since, this is not case, we can find some non-zero semisimple element of L. A Lie subalgebra (which exists by this remark) whose all elements are semisimple will be called *toral subalgebra.*

Lemma 5 *Let T be a toral subalgebra of L. Then T is abelian.*

Proof:
We want to prove that for any $x \in T$, $ad_T x = 0$. Since $ad(x)$ is diagonizable, it suffices to prove that $ad_T x$ has no non-zero eigenvalues. Assume the contrary that $y \in T$ is some non-zero eigenvector corresponding to some non-zero eigenvalue, i.e, $[x, y] = ay$, where $a \in \mathbb{C}$ is not zero. This implies, $ad(y)(x) = -ay$ is eigenvector of y corresponding to eigenvalue 0.

Now write $x = \sum_{i \in I} v_i$ as linear combination of eigenvectors of $ad_T(y)$. Applying $ad_T(y)$ to x, we get $ad(y)(X) = \sum_j a_j v_j$, where v_j are eigenvectors corresponding to non-zero eigenvalues of $ad_T(y)$. But this is not possible since $ad_T(y)(x)$ itself is eigenvector corresponding to eigenvalue zero. Hence, $a = 0$ and our assertion is proved.

QED

Let H be maximal toral subalgebra of L, which will be called *Cartan subalgebra*. Since H is abelian and every element of H is semisimple, its adjoint action on L is simultaneously diagonizable. In other words, L can be written as direct sum of subspaces $L_\alpha = \{x \in L : h.x = \alpha(h)x, \ \forall h \in H\}$, where $\alpha \in H^*$. The set of non-zero $\alpha \in H^*$, such that L_α is non-zero, will be called set of *roots*, denoted by Φ and L_α is called *root space*. We call *Cartan Decomposition*, the following

$$L = L_0 \oplus \left(\bigoplus_{\alpha \in \Phi} L_\alpha \right)$$

Here L_0 is same as centralizer of H in L, denoted by $C_L(H)$. Since H is abelian, $H \subseteq C_L(H)$. Our first aim is to prove that H is exactly equal to its centralizer.

Proposition 1 *For any $\alpha, \beta \in H^*$, we have $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$. If $x \in L_\alpha$, $\alpha \neq 0$, x is ad-nilpotent. Also, if $\alpha, \beta \in H^*$, such that $\alpha + \beta \neq 0$, then L_α is orthogonal to L_β with respect to Killing form.*

Proof:
Let $x \in L_\alpha$ and $y \in L_\beta$. For any $h \in H$, we have the following (using Jacobi identity)

$$ad(h)([x,y]) = [h,[x,y]] = [[h,x],y] + [x,[h,y]] = (\alpha(h) + \beta(h))([x,y])$$

The second assertion follows from first, since L is finite dimensional.

For last part of assertion, choose $h \in H$, such that $(\alpha + \beta)(h) \neq 0$. For any $x \in L_\alpha$ and $y \in L_\beta$, invariance of Killing form yields

$$\kappa([h,x],y) = -\kappa([x,h],y) = -\kappa(x,[h,y])$$

which gives that $(\alpha(h) + \beta(h))\kappa(x,y) = 0$ and hence $\kappa(x,y) = 0$.

QED

Corollary 5 *The restriction of Killing form to $L_0 = C_L(H)$ is non-degenerate.*

Proof:
Since κ is non-degenerate and $\kappa(L_0, L_\alpha) = 0$ by previous proposition, we get the corollary.

QED

Proposition 2 *If H is maximal toral subalgebra of L, then $C_L(H) = H$.*

Proof:
We write $C = C_L(H)$ and proceed in following steps:

C contains semisimple and nilpotent parts of its elements $x \in C$ if and only if $ad(x)$ maps H into 0. Since semisimple and nilpotent parts are just polynomials of given endomorphism, they also satisfy this property. That is, for any $x \in C$, $ad(x)_s$ and $ad(x)_n$ also map H into 0 and hence are in C.

All semisimple elements of C are in H If x is semisimple and lie in C, then $H + \mathbb{C}x$ is again toral subalgebra (since sum of commuting semisimple endomorphisms is again semisimple). By maximality of H, x must lie in H.

The restriction of κ to H is non-degenerate Let $h \in H$ be such that $\kappa(h, H) = 0$. Therefore, for any semisimple elements x of C, we have $\kappa(h, x) = 0$; since by previous step every semisimple element of C is in H. So take some nilpotent element of C, say y. Since h and y commute and $ad(y)$ is nilpotent, we have $Tr(ad(h)ad(y)) = 0$. This proves that $\kappa(h, C) = 0$, which together with non-degeneracy of Killing form on C yields, $h = 0$.

C is nilpotent Let x be any element of C and write $x = x_s + x_n$ via Jordan decomposition. Since $x_s \in H$, $ad_C(x)$ is nilpotent. $ad_C(x_n)$ is nilpotent anyway. So $ad_C(x)$, sum of commuting nilpotent endomorphisms, is nilpotent. By Engel's Theorem, we have C is nilpotent.

$C \cap [H, H] = 0$ Since $[H, C] = 0$ and $\kappa(H, [C, C]) =$ (because of invariance of κ). This together with non-degeneracy of κ on H proves the assertion.

C is abelian If not, then $[C, C] \neq 0$ and since C is nilpotent, by corollary to Theorem 2, we get $Z(C) \cap [C, C] \neq 0$. Assume $z \neq 0$ lies in this intersection. By previous step, z can't be semisimple. So, it has non-zero nilpotent part (say n), which commutes with every element of C (since it is in $Z(C)$) and hence $\kappa(n, C) = 0$, which is absurd.

$C = H$ If not, then C contains some non-zero nilpotent element, say x. Since C is abelian, it must commute with every element of C and hence $\kappa(x, C) \neq 0$. This contradiction proves the proposition.

QED

Corollary 6 *The restriction of Killing form to H is non-degenerate.*

Therefore, we can identify H and H^*, via Killing form. More explicitly saying

$$\begin{array}{ccc} H^* & \longrightarrow & H \\ \phi & \longmapsto & t_\phi \end{array}$$

where $\phi(h) = \kappa(t_\phi, h)$, for every $h \in H$.

3.3 Root Space Decomposition

In this section, we study further properties of root spaces and introduce Chevalley basis of semisimple Lie algebra, and Chevalley involution.

Proposition 3 *1. Φ spans H^*.*

2. If $\alpha \in \Phi$ then $-\alpha \in \Phi$.

3. Let $\alpha \in \Phi$ and $x \in L_\alpha$, $y \in L_{-\alpha}$. Then we have

$$[x, y] = \kappa(x, y)t_\alpha$$

4. If $\alpha \in \Phi$, then $[L_\alpha, L_{-\alpha}]$ is one-dimensional, spanned by t_α.

5. $\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) \neq 0$, for every $\alpha \in \Phi$.

6. If $\alpha \in \Phi$ and $x_\alpha \in L_\alpha$ is some non-zero element, then there exists $y_\alpha \in L_{-\alpha}$, such that $x_\alpha, y_\alpha, h_\alpha = [x_\alpha, y_\alpha]$ span three dimensional simple subalgebra of L isomorphic to $\mathfrak{sl}_2(\mathbb{C})$ via

$$x_\alpha \longmapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; y_\alpha \longmapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \; h_\alpha \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

7. $h_\alpha = \frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}$. And $h_\alpha = -h_{-\alpha}$.

PROOF:
(1) If Φ doesn't span H^*, then we can find a non zero $h \in H$, such that $\alpha(h) = 0$ for every $\alpha \in \Phi$. But this implies that $[h, L_\alpha] = 0$, which combined with the fact that H is abelian implies that $[h, L] = 0$. But since center of L is trivial, we get $h = 0$.

(2) If $\alpha \in \Phi$ but $-\alpha \notin \Phi$, then $\kappa(L_\alpha, L_\beta) = 0$, for every β, which is absurd since κ is non-degenerate.

(3) Let $\alpha \in \Phi$ and $x \in L_\alpha$, $y \in L_{-\alpha}$. For any $h \in H$, we get

$$\kappa(h, [x, y]) = \kappa([h, x], y) = \alpha(h)\kappa(x, y) = \kappa(t_\alpha, h)\kappa(x, y) = \kappa(h, \kappa(x, y)t_\alpha)$$

This says that H is orthogonal to $[x, y] - \kappa(x, y)t_\alpha$, which together with the fact that κ is non-degenerate, implies that $[x, y] = \kappa(x, y)t_\alpha$.

(4) We have already seen that t_α spans $[L_\alpha, L_{-\alpha}]$, provided that later is non-zero. Since κ is non-degenerate, for any $x \in L_\alpha$, non-zero, we can find $y \in L_{-\alpha}$ such that $\kappa(x, y) \neq 0$. This proves part (4).

(5) Suppose $\alpha(t_\alpha) = 0$, so that $[t_\alpha, x] = 0 = [t_\alpha, y]$, for every $x \in L_\alpha$ and $y \in L_{-\alpha}$. Now we can find $x \in L_\alpha$ and $y \in L_{-\alpha}$, such that $\kappa(x, y) \neq 0$. Assume that x, y are selected so that $\kappa(x, y) = 1$ and hence $[x, y] = t_\alpha$. This implies subalgebra of L, spanned by x, y, t_α, say S, is solvable. Therefore, for every $s \in [S, S]$ is nilpotent and hence $ad_L(t_\alpha)$ is both semisimple and nilpotent. This implies $ad_L(t_\alpha) = 0$ and hence $t_\alpha = 0$.

To prove rest of the proposition, it suffices to choose for every $x_\alpha \in L_\alpha$, $y_\alpha \in L_{-\alpha}$, such that $\kappa(x_\alpha, y_\alpha) = \frac{2}{\kappa(t_\alpha, t_\alpha)}$ and set $h_\alpha = 2t_\alpha/\kappa(t_\alpha, t_\alpha)$.

QED

For any $\alpha \in \Phi$, let S_α be subalgebra generated by $x_\alpha, y_\alpha, h_\alpha$, which is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$, as proved above.

Now fix any root $\alpha \in \Phi$. Let M be subalgebra of L generated by roots spaces corresponding to all roots of the form $c\alpha$, where $c \in \mathbb{C}$, and H. S_α acts on this subalgebra M via adjoint representation. The weights of h_α on M are 0 and $2c$. Since these must be integers, we get that c must be some integral multiple of 1/2. Since S_α acts trivially on $Ker(\alpha)$, which is subspace of H of codimension one, complementary to $\mathbb{C}h_\alpha$. Also, S_α is irreducible sub-S_α-module of M. Together $Ker(\alpha)$ and S_α exhaust occurance of weight 0. Therefore, only even weights occuring in M are 0 and ± 2. This proves that twice of a root is not a root. Hence $\alpha/2$ cannot be a root. Therefore, we get that $M = H + S_\alpha$. In particular, $dim(L_\alpha) = 1$ and only multiples of α which occur as root are $\pm\alpha$.

Now set $K = \sum_{i \in \mathbb{Z}} L_{\beta + i\alpha}$, where β is some other root, distinct from $\pm\alpha$. We observe that K is irreducible sub-S_α-module of L. Therefore, highest (resp. lowest), weight must correspond to $\beta(h_\alpha) + 2q$ (resp. $\beta(h_\alpha) - 2r$), if q, r are largest integers such that $\beta + q\alpha$ and $\beta - r\alpha$ are roots. But since for irreducible $\mathfrak{sl}_2(\mathbb{C})$ module, weights from arithmetic progression, with common difference 2, we get that for every i, $-r \leq i \leq q$, $\beta + i\alpha$ is a root. Moreover, $(\beta - r\alpha)(h_\alpha) = -(\beta + q\alpha)(h_\alpha)$ and hence $\beta(h_\alpha) = r - q$. We summarize all this in next proposition:

Proposition 4

1. *$\alpha \in \Phi$ implies that $dim(L_\alpha) = 1$. In particular, $S_\alpha = L_\alpha + L_{-\alpha} + H_\alpha$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$ (where $H_\alpha = [L_\alpha, L_{-\alpha}]$).*
2. *If $\alpha \in \Phi$, then only scalar multiples of α which are roots, are α and $-\alpha$.*
3. *$\alpha, \beta \in \Phi$ implies that $\beta(h_\alpha) \in \mathbb{Z}$ and $\beta - \beta(h_\alpha)\alpha \in \Phi$.*
4. *$\alpha, \beta, \alpha + \beta \in \Phi$ implies that $[L_\alpha, L_\beta] = L_{\alpha+\beta}$.*
5. *Let $\alpha, \beta \in \Phi$ and r, q be largest integers such that $\beta - r\alpha$ and $\beta + q\alpha$ are roots. Then $\beta + i\alpha \in \Phi$ for every i, $-r \leq i \leq q$. Moreover, $\beta(h_\alpha) = r - q$.*
6. *L is generated by L_α; $\alpha \in \Phi$, as Lie algebra.*

References

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