

Tits Systems

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1 Review of Basic Results

First we give definitions and basic properties of Topological Groups, Fibre Bundles and Basic Tits Systems which will be required later. We will omit the proofs which can be found in [1], [2] and [3] respectively.

1.1 Topological Groups

Definition 1 A topological group G , is a set with group structure and a topology such that:

$$\begin{aligned} G \times G &\longrightarrow G \\ (g, h) &\longmapsto g^{-1}h \end{aligned}$$

is continuous.

Definition 2 A topological group G acting on a topological space X is said to act continuously if

$$\begin{aligned} G \times X &\longrightarrow X \\ (g, x) &\longmapsto g.x \end{aligned}$$

is continuous. Moreover, $g.(h.x) = (gh).x$, for every $g, h \in G$ and $x \in X$.

Definition 3 A topological group G acting continuously on a topological space X is said to act properly if

$$\begin{aligned} G \times X &\longrightarrow X \times X \\ (g, x) &\longmapsto (g, g.x) \end{aligned}$$

is a proper map.

Proposition 1 *Let G be a topological group and H be a subgroup of G . Then*
 (a) G/H with quotient topology is Hausdorff if and only if H is closed subgroup of G .
 (b) G/H with quotient topology is discrete if and only if H is open subgroup of G .

Theorem 1 *Let G be topological group and H be a closed subgroup of G . Then H acts properly on G by left translations. Moreover, if we have K , a compact subset of G and A , a closed subset of G , then $K.A$ is closed in G . Finally, if K' is any subset of G , such that $\overline{K'} = K$ is compact in G then for any closed subset A of G , we have*

$$\overline{K'.A} = K.A$$

1.2 Fibre Bundles

Definition 4 Coordinate Bundles

A coordinate bundle \mathfrak{B} consists of tuple $(B, X, p, Y, G, \{V_j, \varphi_j\}_{j \in J})$, where

1. B, X, Y are topological spaces called bundle space, base space and fibre space respectively.
2. $p : B \longrightarrow X$ is continuous function, called projection.
3. $\{V_j\}_{j \in J}$ is open covering of X , called coordinate neighborhoods and for each j

$$\varphi_j : V_j \times Y \longrightarrow p^{-1}(V_j)$$

is homeomorphism, called coordinate functions, such that $p\varphi_j(x, y) = x$, for each $x \in V_j$ and $y \in Y$.

4. G is topological group acting effectively on Y , called the structure group.
5. If we define

$$\varphi_{j,x} : Y \longrightarrow p^{-1}(x)$$

as $\varphi_{j,x}(y) = \varphi_j(x, y)$, we get for each $i, j \in J$ and $x \in V_i \cap V_j$, the following homeomorphism of fibre space

$$\varphi_{j,x}^{-1}\varphi_{i,x} : Y \longrightarrow Y$$

Then it is required that this homeomorphism must be same as action of some element of G . $Y_x := p^{-1}(x)$ is called fibre over x .

6. Moreover, for every $i, j \in J$, the map

$$g_{ji} : V_i \cap V_j \longrightarrow G$$

defined as $g_{ji}(x) = \varphi_{j,x}^{-1} \varphi_{i,x}$ is continuous. These maps are called coordinate transformations.

Definition 5 Bundle Maps

Let $\mathfrak{B}, \mathfrak{B}'$ be two coordinate bundles with same fibre space and structure group. By a bundle map $h : \mathfrak{B} \longrightarrow \mathfrak{B}'$, we mean a continuous function $h : B \longrightarrow B'$ between bundle spaces of respective coordinate bundles such that

- Restriction of h to any fibre Y_x is homeomorphism of Y_x onto some fibre $Y_{x'}$ over $x' \in X'$. Thus there is a induced continuous map $\bar{h} : X \longrightarrow X'$ such that the following diagram commutes.

$$\begin{array}{ccc} B & \xrightarrow{h} & B' \\ \downarrow p & & \downarrow p' \\ X & \xrightarrow{\bar{h}} & X' \end{array}$$

- If $x \in V_j \cap \bar{h}^{-1}(V'_k)$ and h_x denotes the restriction of h to fibre over x , the we get the following homeomorphism of fibre space

$$\bar{g}_{kj} = \varphi'_{k,x'}^{-1} h_x \varphi_{j,x} : Y \longrightarrow Y$$

which should be same as action of some element of G on Y . Moreover, for any two such pairs j, k the map

$$\bar{g}_{kj} : V_j \cap \bar{h}^{-1}(V'_k) \longrightarrow G$$

should be continuous. These functions are called mapping transformations.

Definition 6 Bundle Equivalence

Two coordinate bundles $\mathfrak{B}, \mathfrak{B}'$ with same base space, fibre space and structure group are said to be equivalent if there exists a bundle map $h : \mathfrak{B} \longrightarrow \mathfrak{B}'$ such that the induced map $\bar{h} : X \longrightarrow X$ is identity.

Definition 7 Trivial Bundle or Product Bundle

A coordinate bundle is called product bundle or trivial bundle if there is only one coordinate neighborhood and structure group is trivial. Equivalently saying, there is a continuous function from base space to bundle space $f : X \longrightarrow B$, such that $pf(x) = x$ for every $x \in X$.

Theorem 2 Bundle Structure Theorem

Suppose B is topological group and G is closed subgroup of B . Assume that there is a continuous function defined on some open neighborhood of $\bar{1}$ in B/G , say V to B , such that $pf(x) = x$ for every $x \in V$, where

$$p : B \longrightarrow B/G$$

is canonical map. (f is called local cross-section of G in B) Then there exists a coordinate bundle with B as bundle space B/G as base space, and G , both as structure group and fibre space (such bundles are called principal bundles).

Theorem 3 Any bundle over normal, contractible, locally compact space X , for which every open covering has countable refinement, is equivalent to product bundle.

Definition 8 Induced Bundles or Pullback Bundles

Let \mathfrak{B} be a coordinate bundle. Assume X is a topological space and $\eta : X \longrightarrow X'$ is a continuous function. Then we can define a bundle over X , say \mathfrak{B} , together with bundle map $h : \mathfrak{B} \longrightarrow \mathfrak{B}'$ such that $\bar{h} = \eta$.

Define: $B := \{(x, b') \in X \times B' \ni \eta(x) = p'(b')\}$

and let $p : B \longrightarrow X$ and $h : B \longrightarrow B'$ be canonical projections. Then by definition of B , h induces η on base spaces. To complete the construction of bundle, we need to give coordinate neighborhoods and coordinate functions. Let $\{V_j := \eta^{-1}(V'_j)\}_{j \in J}$ be coordinate neighborhoods, and

$$\varphi_j : V_j \cap Y \longrightarrow p^{-1}(V_j)$$

be defined as $\varphi_j(x, y) = (x, \varphi'_j(\eta(x), y))$. The bundle thus constructed is called induced bundle or pullback bundle and is denoted by $\eta^{-1}\mathfrak{B}'$.

1.3 CW-complexes

In this section we will give the definition of CW complex.

Definition 9 A CW-complex consists of a Hausdorff topological space X called underlying space together with partition of X into collection $\{X_\alpha\}$ of disjoint subsets, called cells, satisfying the following conditions.

(C1) For every α , we have a continuous function, for some $n(\alpha) > 0$,

$$f_\alpha : D^{n(\alpha)} \longrightarrow X$$

which maps interior of disc $D^{n(\alpha)}$ homeomorphically onto X_α . $n(\alpha)$ is called dimension of cell X_α

(C2) For any α , every $x \in \overline{X_\alpha} \setminus X_\alpha$ lies in cell of lower dimension.

(C3) (Closure finiteness) Every point of X lies in finite subcomplex, that is, a closed subset of X , which is finite union of cells.

(C4) (Whitehead Topology) A subset of X is closed if and only if its intersection with every finite subcomplex is closed.

1.4 Coxeter Systems

Assume W is a group generated by $S \subset W$, which consists of elements of order 2. For any $s, s' \in S$, let $m(s, s')$ denote the order of ss' in W .

Definition 10 Coxeter Systems (W, S) is called Coxeter System if set of generators S and set of relations

$$\{(ss')^{m(s,s')} = 1 \ni m(s, s') < \infty\}$$

give complete presentation of W .

Theorem 4 Consider pair (W, S) and let T be set of conjugates of elements of S . Then (W, S) is Coxeter System if and only if following property holds (called Strong Exchange Condition)

If $t \in T$ and $w \in W$ are such that $l(tw) \leq l(w)$, then for any expression $w = s_1 \dots s_p$, $s_i \in S$, there exists j , $1 \leq j \leq p$ such that

$$t = (s_1 \dots s_{j-1})s_j(s_1 \dots s_{j-1})^{-1}$$

Definition 11 Let (W, S) be a Coxeter System and $Y \subset S$. Denote by W_Y , subgroup generated by Y . Then we define

$$W'_Y := \{w \in W \ni l(wy) > l(w) \forall y \in W_Y\}$$

Lemma 1 *Assuming the notations of previous definition, the following holds:*

$$W'_Y = \{w \in W \ni l(wy) = l(w) + l(y) \forall y \in W_Y\}$$

$$W''_Y = \{w \in W \ni l(wy) > l(w) \forall y \in Y\}$$

Definition 12 Bruhat-Chevalley Ordering

Let (W, S) be Coxeter System. For $v, w \in W$, we say $v \leq w$ if there exist $t_1, \dots, t_p \in T$, set of conjugates of elements of S , such that

- $v = t_p \dots t_1 w$
- $l(t_j \dots t_1 w) \leq l(t_{j-1} \dots t_1 w)$ for every j , $1 \leq j \leq p$.

Lemma 2 *Let (W, S) be Coxeter System. Let $v, w \in W$ be any two elements of W . Fix a reduced decomposition $w = s_1 \dots s_p$ of w . Then $v \leq w$ if and only if there exist $1 \leq j_1 < \dots < j_k \leq p$ such that*

$$v = s_{j_1} \dots s_{j_k}$$

Moreover, this sequence can be chosen in such a way that above decomposition of v is reduced.

1.5 General Tits Systems

Definition 13 A Tits System (G, B, N, S) consists of a group G , subgroups B, N and subset $S \subset N/(B \cap N)$ such that

- (BN1) $B \cap N$ is normal in N and S generates $W := N/(B \cap N)$.
- (BN2) B and N generate G .
- (BN3) For any $s \in S$, $sBs^{-1} \not\subset B$.
- (BN4) For any $w \in W$ and $s \in S$, we have

$$C(s).C(w) \subset C(w) \cup C(sw)$$

where $C(w) := B\bar{w}B$, where \bar{w} is any coset representative of w in N .

Definition 14 For any Tits System, (G, B, N, S) , $W := N/(B \cap N)$ is called *Weyl group* associated with Tits System. Any subgroup of G containing B is called *standard parabolic subgroup* of G . And any subgroup of G containing some conjugate of B is called *parabolic subgroup* of G .

The following properties will be useful in study of further topics in Tits Systems.

Theorem 5 Let (G, B, N, S) be a Tits System.

- (a) All elements of S are of order 2.
- (b) For any $Y \subset S$, $P_Y := BW_Y B$ is subgroup of G .
- (c) Bruhat Decomposition

$$G = \bigsqcup_{w \in W} C(w)$$

And for any $Y, Y' \subset S$

$$G = \bigsqcup_{w \in W_Y \backslash W / W_{Y'}} P_Y w P_{Y'}$$

- (d)

$$C(s).C(w) = \begin{cases} C(sw); & l(sw) \geq l(w) \\ C(w) \cup C(sw); & l(sw) \leq l(w) \end{cases}$$
- (e) (W, S) is a Coxeter System.
- (f) For any reduced decomposition, $w = s_1 \dots s_p$, we have

$$C(s_i) \subset \langle C(w) \rangle$$

for every i , $1 \leq i \leq p$. Moreover,

$$\langle B, wBw^{-1} \rangle = \langle C(w) \rangle$$

- (g) For any $Y, Y' \subset S$, $P_Y = P_{Y'}$ if and only if $Y = Y'$. Moreover for any standard parabolic subgroup P of G , there is some $Y \subset S$, such that $P = P_Y$. That is, there are precisely $2^{|S|}$ standard parabolic subgroups of G .

(h) Every parabolic subgroup P of G is proper normalizer. Two standard parabolic subgroups of G are conjugate if and only if they are equal.

The following lemma will be extensively used in study of Topological Tits Systems, and we will give its proof here.

Lemma 3 Let (G, B, N, S) be a Tits System. Let $Y \subset S$ and $w \in W'_Y$. For any decomposition $w = w_1 \dots w_p$, such that

$$l(w) = l(w_1) + \dots + l(w_p)$$

following holds:

(a) Let A_i be any subset of $C(w_i)$, such that $\varpi_i : A_i \rightarrow C(w_i)/B$ is surjective (resp. bijective), for every i , $1 \leq i \leq p$. Then

$$\begin{aligned} \varphi : A_1 \times \dots \times A_p &\longrightarrow BwP_Y/P_Y \\ (a_1, \dots, a_p) &\longmapsto a_1 \dots a_p \pmod{P_Y} \end{aligned}$$

is surjective (resp. bijective)

(b) Assume, in addition, that each $w_i \in S$ and let $Z_i \subset P_{w_i}$ be any subset containing e , such that the canonical projection $Z_i \hookrightarrow P_{w_i} \rightarrow P_{w_i}/B$ is surjective. Then the image of map

$$\psi : Z_1 \times \dots \times Z_p \longrightarrow G/P_Y$$

is precisely equal to

$$\bigcup_{v \leq w} BvP_Y/P_Y$$

where \leq is Bruhat-Chevalley ordering on Coxeter Group W .

PROOF:

(a) We first prove surjectivity of φ assuming that $A_i \rightarrow C(w_i)/B$ is surjective, for every i .

$$\text{Im } \varphi = C(w_1) \dots C(w_p)P_Y \pmod{P_Y}$$

which is same as $C(W)P_Y \pmod{P_Y}$ and hence φ is surjective.

Now assume that each $\pi_i : A_i \rightarrow C(w_i)/B$ is bijective. We will prove that φ is injective in this case. Assume that

$$\varphi(a_1, \dots, a_p) = \varphi(a'_1, \dots, a'_p)$$

where $a_i, a'_i \in C(w_i)$. This is same as saying that $a_1 \dots a_p = a'_1 \dots a'_p p'$ for some $p' \in C(v)$, where $v \in W_Y$. Now as $w \in W'_Y$, $C(w).C(v) = C(wv)$, which implies

$$a_1 \dots a_p = a'_1 \dots a'_p p' \in C(w) \cap C(wv)$$

and hence $v = 1$ and $p' \in B$. Let $w_1 = s_1 \dots s_n$ be reduced decomposition of w_1 . For each j , $1 \leq j \leq n$, let $D_j \subset C(s_j)$ be chosen such that $D_j \longrightarrow C(s_j)/B$ is bijective map. Thus we can write

$$a_1 = d_1 \dots d_n b \quad a'_1 = d'_1 \dots d'_n b'$$

where $d_i, d'_i \in D_i$ and $b, b' \in B$.

$$d_1^{-1} d_1 \dots d_n b a_2 \dots a_p = d'_1 \dots d'_n b' a'_2 \dots a'_p p' \quad (1)$$

Now $d_1, d'_1 \in C(s_1)$. Therefore, $d_1^{-1} d_1 \in C(s_1) \cup B$. If we assume $d_1^{-1} d_1 \in C(s_1)$, we get that both sides of (1) belong to $C(w) \cap C(s_1 w) = \phi$. Hence $d_1^{-1} d_1 \in B$ and therefore they are equal because $D_1 \longrightarrow C(s_1)/B$ is bijective. Similarly we can prove that

$$d'_2 = d_2, \dots, d'_n b' = d_n b$$

which again follows from bijectivity of $D_n \longrightarrow C(s_n)/B$ that $d_n = d'_n$. This implies, because of bijectivity of $A_1 \longrightarrow C(w_1)/B$ that $a_1 = a'_1$. Following similar way of proof, we can prove that

$$a_2 = a'_2, \dots, a_p = a'_p p'$$

and as $p' \in B$, we get $a_p = a'_p$, proving injectivity of φ .

(b) Now assume that $Z_i = \{1\} \cup A_i$ such that $A_i \subset C(w_i)$ and $A_i \longrightarrow C(w_i)/B$ is surjective. Then

$$Im\psi = \bigcup_{(1 \leq i_1 < \dots < i_k \leq p)} C(w_{i_1}) \dots C(w_{i_k}) P_Y \text{ mod } P_Y$$

Now from characterization of Bruhat-Chevalley ordering, as stated in Lemma 2, following is immediate

$$\bigcup_{(1 \leq i_1 < \dots < i_k \leq p)} C(w_{i_1} \dots w_{i_k}) = \bigcup_{(1 \leq i_1 < \dots < i_k \leq p)} C(w_{i_1}) \dots C(w_{i_k}) = \bigcup_{v \leq w} C(v)$$

and hence the lemma follows.

QED

2 Topological Tits Systems

Definition 15 Let (G, B, N, S) be a Tits System. We say it is *Topological Tits System* if following additional conditions hold:

- (BN5) G is Hausdorff Topological Group, B, N are closed subgroups of G . $B \cap N$ is open in N , and hence the Weyl group W has discrete topology.
- (BN6) For each $s \in S$, P_s/B with quotient topology is homeomorphic to $S^{n(s)}$, sphere of dimension $n(s) > 0$. Moreover, the canonical projection $\pi_s : P_s \rightarrow P_s/B$ has local continuous cross-section and hence has principal B -bundle structure. (cf. Theorem 2)

Theorem 6 Let (G, B, N, S) be Topological Tits System and $Y \subset S$. Then we have

- (a) Assume P_Y is closed in G , then for any $w \in W'_Y$

$$\overline{BwP_Y} = \bigsqcup_{\substack{v \leq w \\ v \in W'_Y}} BvP_Y$$

In particular B -orbits in G/P_Y are locally closed.

- (b) P_Y is closed in G if W_Y is finite.

- (c) Let P_Y be closed in G . Define the following topology on G/P_Y :

Inductive Limit Topology: $A \subset G/P_Y$ is closed if and only if $\pi_Y^{-1}(A) \cap \overline{BwP_Y}$ is closed in G , for every $w \in W'_Y$, where $\pi_Y : G \rightarrow G/P_Y$ is canonical map.

With this topology, G/P_Y is CW-complex with cells $\{BwP_Y\}_{w \in W'_Y}$

PROOF:

Claim: There exists continuous map $\sigma_s : D^{n(s)} \rightarrow P_s$ for each $s \in S$ such that the following diagram commutes:

$$\begin{array}{ccc} D^{n(s)} & \xrightarrow{\sigma_s} & P_s \\ & \searrow & \downarrow \pi_s \\ & & P_s/B \cong S^{n(s)} \end{array}$$

where D^n is closed unit disc in \mathbb{R}^n . We will denote boundary of this disc by $\partial(D^n)$ and interior of the disc by \dot{D}^n , i.e, $\partial(D^n) = D^n \setminus \dot{D}^n$. Then we recall that $\partial(D^n) = S^{n-1}$ and $D^n/\partial(D^n) \cong S^n$. Therefore, $P_s/B \cong S^n$ implies that there is a continuous map

$$\alpha_s : (D^{n(s)}, \partial(D^{n(s)})) \longrightarrow (P_s/B, B/B)$$

That is, continuous map from $D^{n(s)}$ to P_s/B , which maps the boundary identically to identity coset.

Now we know that $\pi_s : P_s \longrightarrow P_s/B$ has principal B -bundle structure, therefore, we can talk about α_s -induced bundle over $D^{n(s)}$ such that following diagram commutes.

$$\begin{array}{ccc} \alpha_s^{-1}(P_s) & \xrightarrow{h} & P_s \\ \downarrow p & & \downarrow \pi_s \\ D^{n(s)} & \xrightarrow{\alpha_s} & P_s/B \end{array}$$

Referring to Theorem 3, we know that bundle over $D^{n(s)}$ is equivalent to trivial bundle and hence there is a map $f : D^{n(s)} \longrightarrow \alpha_s^{-1}(P_s)$ such that $pf(x) = x$, for every $x \in D^{n(s)}$. Therefore, we can define continuous map from $D^{n(s)}$ to P_s as $\sigma_s = h \circ f$.

$$\pi_s \circ \sigma_s = \pi_s \circ h \circ f = \alpha_s \circ p \circ f = \alpha_s$$

completing the proof of claim.

(a) Now we come to the proof of part (a). Assuming the notations we introduced in above claim we have following commuting diagram, for every $s \in S$.

$$\begin{array}{ccc} D^{n(s)} & \xrightarrow{\sigma_s} & P_s \\ & \searrow \alpha_s & \downarrow \pi_s \\ & & P_s/B \end{array}$$

Moreover, α_s is surjective. Because $\alpha_s|_{D^{n(s)}}$ is homeomorphism of interior of disc onto $(P_s \setminus B)/B$ and $\alpha_s(\partial(D^{n(s)})) = B/B$, by definition of α_s .

Denote by A_s , the image of σ_s in P_s . Being image of compact set it is compact and we can further assume that $1 \in A_s$. By surjectivity of α_s , it follows that

$$\pi_s(A_s) = P_s/B = A_s B \tag{2}$$

$$\overline{A_s \cap C(s)} = A_s \quad (3)$$

As σ_s carries interior of disc into $C(s)$ and boundary into B . Denote by $\hat{A}_s := A_s \cap C(s)$.

Assuming the notations of statement of part (a), let $w = s_1 \dots s_p$ is reduced decomposition of w . For each i , we have the canonical map $A_{s_i} \rightarrow P_{s_i}/B$ is surjective and hence we can apply Lemma 3, to get that

$$A_{s_1} \dots A_{s_p} P_Y = \bigcup_{v \leq w} BvP_Y = \bigsqcup_{\substack{v \leq w \\ v \in W'_Y}} BvP_Y \quad (4)$$

Also, as A_{s_i} is compact and P_Y is closed in G , from Theorem 1, we get that $A_{s_1} \dots A_{s_p} P_Y$ is closed in G and

$$\overline{\hat{A}_{s_1} \dots \hat{A}_{s_p} P_Y} = A_{s_1} \dots A_{s_p} P_Y \quad (5)$$

Finally, we have that $\hat{A}_{s_i} \cdot B = P_{s_i} \cap C(s_i) = C(s_i)$ and $C(w) = C(s_1) \dots C(s_p)$. Hence we get

$$\hat{A}_{s_1} \dots \hat{A}_{s_p} P_Y = BwP_Y \quad (6)$$

From (4), (5) and (6), we get the result we intended to prove, i.e.,

$$\overline{BwP_Y} = \bigsqcup_{\substack{v \leq w \\ v \in W'_Y}} BvP_Y$$

which directly implies that

$$\overline{BwP_Y} \setminus BwP_Y = \bigsqcup_{\substack{v < w \\ v \in W'_Y}} BvP_Y$$

which is closed in G and hence BwP_Y is locally closed, completing the proof of part (a).

(b) Now assume that W_Y is finite. Then we have

$$P_Y = \bigsqcup_{w \in W_Y} C(w)$$

and the union is finite. Moreover if we substitute $Y = \phi$ in part (a), we get that

$$\overline{C(w)} = \bigsqcup_{v \leq w} C(v)$$

Also, $v \leq w$ implies that $v \in W_Y$. Hence it follows that

$$P_Y = \bigcup_{w \in W_Y} \overline{C(w)}$$

which being a finite union of closed sets is closed in G .

(c) Assume that P_Y is closed, then we will prove that for each $w \in W'_Y$, BwP_Y/P_Y is a cell of dimension $d(w) = \sum_{i=1}^p n(s_i)$, where, $w = s_1 \dots s_p$ is any reduced decomposition of w .

Sticking to the notations of the claim, we define

$$\begin{aligned} \sigma_w : D^{n(s_1)} \times \dots \times D^{n(s_p)} &\longrightarrow G/P_Y \\ (t_1, \dots, t_p) &\longmapsto \sigma_{s_1}(t_1) \dots \sigma_{s_p}(t_p) \text{ mod } P_Y \end{aligned}$$

Then the following are immediate

$$Im(\sigma_w) = (\overline{BwP_Y})/P_Y \tag{7}$$

$$\sigma_w(\partial(D)) \subseteq \bigcup_{\substack{v \leq w \\ v \in W'_Y}} BvP_Y/P_Y \tag{8}$$

$$\sigma_w(\dot{D}) = BwP_Y/P_Y \tag{9}$$

where $D = D^{n(s_1)} \times \dots \times D^{n(s_p)}$ is disc of dimension $d(w)$.

Also σ_w is continuous under inductive limit topology on G/P_Y and

$$\sigma_w|_{\dot{D}} : \dot{D} \longrightarrow BwP_Y/P_Y$$

is bijective continuous map. We will prove that it is homeomorphism.

Assume that C is any closed set in \dot{D} , then

$$\sigma_w|_{\dot{D}}(C) = \sigma_w(\overline{C}) \cap BwP_Y/P_Y$$

which is closed in BwP_Y/P_Y . Thus, $\sigma_w|_D$ has continuous inverse and hence is homeomorphism.

Moreover (8) implies that

$$(\overline{BwP_Y}/P_Y) \setminus BwP_Y/P_Y \subseteq \bigcup_{\substack{v < w \\ v \in W'_Y}} BvP_Y/P_Y$$

which are cells of lower dimension (as $v < w \Rightarrow d(v) < d(w)$)

Closure finiteness follows from (a) and Inductive limit topology on G/P_Y satisfies condition of Whitehead Topology. Hence, G/P_Y is CW-complex with cells $\{BwP_Y/P_Y\}_{w \in W'_Y}$, completing the proof of Theorem.

QED

3 Amalgamated Product of System of groups

Definition 16 Let $\{M_i\}_{i \in I}$ be a family of groups, such that for each $i, j \in I$, $M_{ij} = M_i \cap M_j = M_{ji}$ is a subgroup of both M_i and M_j and has same group structure as subgroup of either of M_i and M_j . Let $\varphi_{ij} : M_{ij} \hookrightarrow M_i$ be canonical injection. Such a collection will be called *system of groups*. By amalgamated product of such system of groups we mean $(M, \{\varphi_i\}_{i \in I})$, where M is a group, and following holds:

- (a) For each $i \in I$, $\varphi_i : M_i \longrightarrow M$ is group homomorphism such that following diagram commutes for every $i, j \in I$

$$\begin{array}{ccc} M_{ij} = M_{ji} & \xrightarrow{\varphi_{ij}} & M_i \\ \downarrow \varphi_{ji} & & \downarrow \varphi_i \\ M_j & \xrightarrow{\varphi_j} & M \end{array}$$

- (b) *Universal Property*: If (H, ψ_i) is any other group together with group homomorphisms satisfying (a), then there exists unique group homomorphism $\psi : M \longrightarrow H$ such that for each $i \in I$ following diagram commutes

$$\begin{array}{ccc} M_i & \xrightarrow{\varphi_i} & M \\ & \searrow \psi_i & \downarrow \psi \\ & & H \end{array}$$

Theorem 7 *Let (G, B, N, S) be a Tits System. Then G is amalgamated product of system of subgroups $\{N, P_s; s \in S\}$.*

PROOF:

It is trivial to verify that $\{N, P_s; s \in S\}$ is a system of subgroups of G . We will assume that G' is amalgamated product of this system together with group homomorphisms $\varphi : N \longrightarrow G'$ and $\varphi_s : P_s \longrightarrow G'$, such that following diagrams are commutative.

$$\begin{array}{ccc} B^c & \longrightarrow & P_s \\ \downarrow & & \downarrow \varphi_s \\ P_{s'} & \xrightarrow{\varphi_{s'}} & G' \end{array} \quad (10)$$

$$\begin{array}{ccc} T \sqcup_s T^c & \longrightarrow & N \\ \downarrow & & \downarrow \varphi \\ P_s & \xrightarrow{\varphi_s} & G' \end{array} \quad (11)$$

Then by Universal Property of amalgamated products, we have a unique group homomorphism $\psi : G' \longrightarrow G$, such that for every $s \in S$, the following diagrams are commuting.

$$\begin{array}{ccc} N & \xrightarrow{\varphi} & G' \\ \searrow \iota_n & & \downarrow \psi \\ & & G \end{array} \quad (12)$$

$$\begin{array}{ccc} P_s & \xrightarrow{\varphi_s} & G' \\ \searrow \iota_s & & \downarrow \psi \\ & & G \end{array} \quad (13)$$

where ι_n and ι_s are canonical injective maps. Denote by β , the restriction of φ_s to B , for any $s \in S$, i.e, $\beta = \varphi_s|_B$ for every $s \in S$. We intend to prove that ψ is an isomorphism in the following steps.

Step 1 G' is generated by $\varphi(N)$ and $\beta(B)$. As G' is generated by $\varphi(N) \cup (\bigcup_{s \in S} \varphi_s(P_s))$ and $P_s = B \cup C(s)$, it is immediately verified that G' is generated by images of N and B under φ and β respectively. Hence ψ

is surjective

Step 2 $G' = \beta(B)\varphi(N)\beta(B)$ We first claim that if $n \in N, b \in B$ are such that $nb n^{-1} \in P_s$, then

$$\varphi_s(nbn^{-1}) = \varphi(n)\beta(b)\varphi(n^{-1})$$

We will prove the claim by induction on length of $\pi(n)$, where $\pi : N \rightarrow W$ is canonical map. For base case, $l(\pi(n)) = 0$ implies $n \in B \cap N$ and the result follows immediately from commutativity of 11.

Now let $w = \pi(n)$. There are two possibilities

(i) $l(sw) \geq l(w)$

In this case we get that

$$nBn^{-1} \cap C(s) \subset (C(w) \cap C(s).C(w)).n^{-1} = \phi$$

Therefore, $nBn^{-1} \cap P_s \subset B$, which implies that $nb n^{-1} \in B$. Therefore, if we choose $s' \in S$, such that $l(s'w) < l(w)$ and $m' \in \pi^{-1}(s')$ and $m \in \pi^{-1}(s'w)$, such that $n = m'm$, we get the following:

$$\begin{aligned} \varphi_s(nbn^{-1}) &= \varphi_s(m'mbm^{-1}m'^{-1}) = \varphi_{s'}(m')\varphi_{s'}(mbm^{-1})\varphi_{s'}(m'^{-1}) \\ &= \varphi(m')\varphi(m)\beta(b)\varphi(m^{-1})\varphi(m'^{-1}) = \varphi(n)\beta(b)\varphi(n^{-1}) \end{aligned}$$

and hence the claim.

(ii) $l(sw) \leq l(w)$

Again writing $n = mm'$, where $m \in \pi^{-1}(s)$ and $m' \in \pi^{-1}(sw)$, we get that

$$\varphi_s(nbn^{-1}) = \varphi_s(mm'bm'^{-1}m^{-1}) = \varphi_s(m)\varphi_s(m'bm'^{-1})\varphi(m^{-1})$$

which is same as $\varphi(n)\beta(b)\varphi(n^{-1})$. Hence in either case, our claim holds.

Now to complete the proof of Step 2, it is enough to show that if $s \in S$ and $m \in \pi^{-1}(s)$, $n \in N$ and $b \in B$, then

$$\varphi(m)\beta(b)\varphi(n) \in \beta(B)\varphi(N)\beta(B)$$

as G' is generated by $\beta(B)$ and $\varphi(N)$.

Let $w = \pi(n)$. As we have $C(s).C(w) \subseteq C(w) \cup C(sw)$, we consider the two possibilities separately that

(i) $mbn \in BnB$

In this case let $mbn = b'nb''$, which implies that $nb''n^{-1} \in P_s$

(ii) $mbn \in BmnB$

In this case let $mbn = b'mnb''$, which together with $C(s).C(s) = B \cup C(s)$, implies that $nb''n^{-1} \in P_s$.

This coupled with first claim gives the following

$$\varphi(m)\beta(b) = \varphi_s(mb) = \begin{cases} \varphi_s(b'nb''n^{-1}) & \text{if } mbn \in BnB \\ \varphi_s(b'mnb''n^{-1}) & \text{if } mbn \in BmnB \end{cases}$$

Hence, we get $\varphi(m)\beta(b)\varphi(n) = \beta(b')\varphi(m'n)\beta(b'')$, where $m' = 1$ or m , which completes the proof of Step 2.

Step 3 ψ is injective.

From Bruhat Decomposition, we have

$$G = BNB$$

Moreover $\psi \circ \beta$ and $\psi \circ \varphi$ are canonical inclusions of B and N into G , because of commutativity of 12 and 13.

This implies that $\text{Ker}\psi \subset \beta(B)$. Moreover

$$\beta(b) \in \text{Ker}\psi \Rightarrow b = \psi(\beta(b)) = 1$$

and hence the injectivity of ψ follows.

QED

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