MOMENT MAP

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In this article I will define a moment map and prove some basic properties of the moment map.

1. Basic set up

This section contains fairly standard notions related to symplectic geometry. Reader can refer to [2], [4] or any other standard text on symplectic geometry for more details.

Let M of a smooth manifold and ω be closed, non-degenerate 2-form on M. The pair (M, ω) is called a symplectic manifold. Similar to last section, ω defines a map from $C^\infty M$ to $Vect(M)$ by:

$$\omega(X_f, .) = df(.) \text{ or equivalently } \iota_{X_f}\omega = df$$

where ι_X denotes truncation operation (of degree -1) on differential forms, corresponding to vector field X. The vector field of the form X_f is called Hamiltonian vector field.

Let us define a bilinear pairing $\{.,.\} : C^\infty M \times C^\infty M \to C^\infty M$ by:

$$\{f, g\} = X_f(g) = \omega(X_f, X_g)$$

Then we have the following properties:

(1) $\{f, f\} = 0$. This is clear since $\omega(X_f, X_f) = 0$.
(2) $\{f, gh\} = \{f, g\}h + \{f, h\}g$. This again is clear since X_f is a derivation and hence satisfies Leibnitz identity.
(3) $X_{\{f,g\}} = [X_f, X_g]$. This is consequence of the following well known formulae from differential geometry:

$$\iota_{[X,Y]} = \mathcal{L}_X\iota_Y - \iota_Y\mathcal{L}_X$$

where $\mathcal{L}_X$ denotes the Lie derivative with respect to X. The following is Cartan formula:

$$\mathcal{L}_X = d\iota_X + \iota_X d$$

Using this we have the following:

$$\begin{aligned}
\iota_{[X_f,X_g]}\omega &= \mathcal{L}_{X_f}\iota_{X_g}\omega - \iota_{X_g}\mathcal{L}_{X_f}\omega \\
&= d\iota_{X_f}\iota_{X_g}\omega + \iota_{X_f}d\iota_{X_g}\omega - \iota_{X_g}d\iota_{X_f}\omega - \iota_{X_g}\iota_{X_f}d\omega \\
&= d\iota_{X_f}\iota_{X_g}\omega \\
&= d\omega(X_f, X_g) = d(\{f, g\})
\end{aligned}$$

By definition this implies the required equation.

(4) $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$. The proof of this uses above properties as:

$$\{f,\{g,h\}\} = -X_{\{g,h\}}f = -[X_g, X_h].f = -X_gX_hf + X_hX_gf = -\{g,\{h,f\}\} - \{h,\{f,g\}\}$$

A commutative algebra A together with $\{.,.\} : A \times A \to A$ satisfying (1), (2) and (4) properties above is called *Poisson algebra*. A manifold M together with a Poisson structure on $C^\infty M$ is called a *Poisson manifold*. We have just shown that every symplectic manifold is Poisson. (Note: the converse is not true). Moreover the property (3) implies that $f \longmapsto X_f$ is Lie algebra homomorphism.

2. Moment map

This section contains the definition of moment map. See [2] for more details.

Assume a connected Lie group G acts effectively on symplectic manifold (M, ω) preserving the symplectic form ω. Then the infinitesimal action is the Lie algebra homomorphism $\mathfrak{g} \to Vect(M)$, say $\xi \longmapsto X_\xi$, given by:

$$X_\xi m := \frac{d}{dt} \exp t\xi.m|_{t=0}$$

Moreover the requirement that G preserves ω implies that

$$\mathcal{L}_{X_\xi}\omega = 0, \quad \text{for every } \xi \in \mathfrak{g}$$

Thus we have the following:

$$\begin{array}{ccc} & & \mathfrak{g} \\ & & \downarrow \\ C^\infty M & \longrightarrow & Vect(M) \end{array}$$

Assumption 1 X_ξ is Hamiltonian for every $\xi \in \mathfrak{g}$. In other words, we know that $\mathcal{L}_{X_\xi}\omega = 0$. Using Cartan's formula and the fact that ω is closed, we get

$$d\iota_{X_\xi}\omega = 0$$

Assumption 1 requires that each of the 1 form $\iota_{X_\xi}\omega$ is exact (which is only known to be closed by the equation above). Hence there exists $f_\xi \in C^\infty M$ satisfying

$$\iota_{X_\xi}\omega = df_\xi$$

Note that these functions are defined only up to a constant. We would like the map $\xi \longmapsto f_\xi$ to be a homomorphism of Lie algebras. One can always choose a basis of $\mathfrak{g}$ and select $f_.$ corresponding to the basis elements and extend the map linearly. The only problem is to preserve the Lie bracket. To this end, assume $\{f_\xi\}_{\xi\in\mathfrak{g}}$ be chosen to satisfy linearity and the previous equation. Note that

$$d(f_{[\xi,\eta]} - \{f_\xi, f_\eta\}) = 0$$

and hence the difference is a constant $c(\xi, \eta)$. Hence we have a map

$$c : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$$

which satisfies: $c(\xi, \xi) = 0$ and $c(\xi, [\eta, \zeta]) +$ cyclic terms $= 0$.

Assumption 2 c is a coboundary, i.e, there exists $\lambda \in \mathfrak{g}^*$ such that $c(\xi, \eta) = \lambda([\xi, \eta])$ for every $\xi, \eta \in \mathfrak{g}$. In terms of Lie algebra cohomology (see [4] for properties of Lie algebra cohomology) the previous property of c asserts that c is a cocycle and hence determines a class in $H^2(\mathfrak{g};\mathbb{R})$. Our assumption is that $[c]=0$.

Now we can use this linear form λ to correct the definition of $f'_\xi s$, namely define

$$\overline{f_\xi} := f_\xi - \lambda(\xi)$$

Then we have

$$\overline{f_{[\xi,\eta]}} - \{\overline{f_\xi}, \overline{f_\eta}\} = c_{\xi,\eta} - \lambda([\xi,\eta]) = 0$$

Hence the map $\mathfrak{g} \to C^\infty M$ is a Lie algebra homomorphism. Define the moment map:

$$\mu : M \to \mathfrak{g}^*$$

by requiring $\langle \mu(m), \xi\rangle = f_\xi(m)$.

3. Properties of the moment map

In this section we prove the basic properties of the moment map. In particular we shall see the following results:

(1) μ is G-equivariant.
(2) If $H \in C^\infty M$ is G invariant function and $\xi \in \mathfrak{g}$, then $\{H, \mu\circ < ., \xi >\} = 0$. In other words, μ gives conserved quantities.
(3) μ is a Poisson morphism (where $\mathfrak{g}^*$ is equipped with Kostant-Kirillov-Souriau bracket).

Of course there are other important theorems related to moment map not treated here, e.g, Atiyah-Guillemin-Sternberg convexity theorem, torus fibrations induced by moment maps, relations to toric varieties and relation to geometric invariant theory. They will be provided in separate article. Reader is referred to the excellent text by N. Chriss and V. Ginzburg [3] for applications of the moment map to representation theory.

3.1. **Noether's theorem.** One of the most important properties of the moment map is that it provides conserved quantities. See [1] for applications of the moment map to classical mechanics.

The moment map is easily seen to be G-equivariant. In other words, we have

$$\mu(g.m) = g.\mu(m) \text{ for every } g \in G$$

where the action on the right hand side is coadjoint action of G on $\mathfrak{g}^*$. In other words we have

$$\langle \mu(g.m), \xi\rangle = \langle \mu(m), g.\xi\rangle$$

We will prove the infinitesimal version of this, namely for every $\xi \in \mathfrak{g}$ we have

$$\frac{d}{dt}|_{t=0}\langle \mu(\exp t\xi.m), \eta\rangle = \frac{d}{dt}|_{t=0} f_\eta(\exp t\xi.m) = (X_\xi f_\eta)(m)$$

Hence we obtain $X_\xi.\langle \mu, \eta\rangle = \langle \mu, [\xi,\eta]\rangle$.

Moreover the derivative of the moment map at a point $m \in M$ is dual to the following:

$$d\mu(m)^t : \mathfrak{g} \to T^*_m M$$

given by $\xi \longmapsto \iota_{X_\xi}\omega(m)$.

Now assume that G action on M preserves a smooth function H. The aim of this section is to prove the following

$$\{H, \mu\} = 0$$

More precisely, we need to show that for every $\xi \in \mathfrak{g}$ we have

$$X_H(\langle \mu, \xi \rangle) = 0$$

This result is usually stated as: μ *is constant along flow of* X_H. Let $\gamma(t)$ be flow along X_H at a point m. Then evaluation of $X_H(\langle \mu, \xi \rangle)$ at m is same as:

$$\frac{d}{dt}|_{t=0}\langle \mu(\gamma(t)), \xi \rangle = \langle d\mu(m)(X_H(m)), \xi \rangle$$

which is same as

$$\langle X_H(m), d\mu(m)^t \xi \rangle = \langle X_H(m), \iota_{X_\xi}\omega(m) \rangle$$

By definition the right hand side equals $\omega(X_\xi, X_H)$ evaluated at m. This is same as (by definition of Hamiltonian vector fields):

$$\omega(X_\xi, X_H) = -dH(X_\xi) = -X_\xi H = 0$$

3.2. **Kostant's theorem.** Now we will prove that μ is a Poisson morphism. Let me recall the Poisson structure on $\mathfrak{g}^*$ (the famous Kirillov-Kostant-Souriau structure): for $f, g \in C^\infty \mathfrak{g}^*$ and $x \in \mathfrak{g}^*$ define

$$\{f, g\}(x) = \langle [df(x), dg(x)], x \rangle$$

where $df(x) : T_x\mathfrak{g}^* = \mathfrak{g}^* \to \mathbb{R}$ is an element of $\mathfrak{g}^{**} = \mathfrak{g}$.

Thus we have to show that for any two functions $f, g \in C^\infty \mathfrak{g}^*$:

$$\{f \circ \mu, g \circ \mu\}_M = \{f, g\}_{\mathfrak{g}^*} \circ \mu$$

Let us first assume that both f and g are linear: that is, there exist $\xi, \eta \in \mathfrak{g}$ such that $f =< ., \xi >$ and $g =< ., \eta >$. Then by definition the vector field $X_{f\mu} = X_\xi$ and hence left hand side becomes:

$$\{f_\xi, f_\eta\}(m)$$

On the other hand the right hand side is:

$$\{f, g\}(\mu(m)) = \langle [\xi, \eta], \mu(m) \rangle = f_{[\xi,\eta]}(m)$$

hence both sides are equal because $\{f_\xi, f_\eta\} = f_{[\xi,\eta]}$ by previous section.

Finally using Leibniz rule the equality, as claimed, holds for polynomial functions on $\mathfrak{g}^*$ and by density of polynomial functions in $C^\infty \mathfrak{g}^*$, the equality holds for all smooth functions.

References

[1] V.I. Arnold, *Mathematical methods of classical mechanics*, Springer-Verlag, 1997.
[2] Michéle Audin, *Torus actions on symplectic manifolds*, Birkhäuser, 2004.
[3] N. Chriss and V. Ginzburg, *Representation theory and complex geometry*, Springer, 1997.
[4] V. Guillen and S. Sternberg, *Symplectic techniques in physics*, Cambridge University Press, 1984.