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CHEVALLEY: SOME REMINISCENCES

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I learnt my algebraic geometry first from Chevalley. I owe a lot to my apprenticeship with Chevalley when I participated in his seminar on Picard varieties in 1958–59.

1. In 1957 I had already spent four years as a research scholar at the Tata Institute of Fundamental Research (TIFR). At the initiative of K. Chandrasekharan who was in charge of its School of Mathematics at that time, I was sent to work in Paris towards the beginning of the academic year 1957–58. By that time I had submitted my thesis to the University of Bombay. However, my training was not focussed and I had no idea as to which area should be my speciality.

I looked forward to attending and imbibing the Cartan seminar. That year it was on "Fonctions Automorphes". Several well-known mathematicians were present at the seminar. The first talk was given by A. Weil. Most of the mathematics went over my head, though I religiously continued to attend the talks. That year the Chevalley Seminar was a continuation of the one of the previous year on "Anneaux de Chow et Applications". Here again I understood very little of what was talked about. One exception was a course of Chevalley on "Fondements de la Géometrie Algébrique" (cf. [C1]) which I found that I could follow as it was self-contained. I concentrated working on this course. Towards the end of 1957, I succeeded in proving that a projective module over a polynomial ring in two variables is free. In fact, if I remember right I related this to Serre when I met him for the first time in December 1957. Thereafter, I had the benefit of talking to Serre on several occasions. I followed his course on "Multiplicités et Intersections". All these developments were encouraging but I was yet to find my major field of research.

Chevalley announced a seminar on Picard Varieties in 1958–59 and at the first meeting of the seminar H. Cartan chose me as one of the speakers. This turned out to be a turning point in my career. The main part of the seminar

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was devoted to the construction of the Picard variety of what is called a *semi-complete* normal variety. I gave many talks following the personal notes of Chevalley. Chevalley often invited me to his house for discussions concerning the seminar. There I had the occasion to meet Mrs. Chevalley, as well as his daughter. Sometimes I met also Japanese mathematicians with whom Chevalley engaged in playing the game "Go".

Chevalley seemed quite pleased with my understanding and presentation of his ideas in my talks, though they were certainly not models of presentation. In the course of the seminar, he suggested to me the problem of constructing the Picard variety of an arbitrary complete variety (which need not be normal). I solved this problem during the summer of 1959. The seminar continued for a little while in 1959–60, where I presented my solution. Then, for the written version of the seminar, he asked me to write up my work. His construction of the Picard variety of a semi-complete normal variety, given at the seminar, was later published in his paper "Sur la théorie de la variété de Picard" (cf. [C2]).

It was indeed a great period to be in Paris. The Grothendieck revolution in algebraic geometry was just starting. I distinctly remember Serre remarking during a chance meeting near the Pantheon sometime in the beginning of 1958–59 that Grothendieck was planning to rewrite algebraic geometry. Chevalley was one of the few among the older generation to understand and appreciate Grothendieck's work. Once he remarked to me that Grothendieck had advanced algebraic geometry by fifty years and that he understood the importance of the language of schemes because of some of his own work. It is clear that he was referring to all his work suggesting the possibility of constructing semi-simple group schemes over $\mathbb{Z}$. One knows that he did this later (cf. [C4]) and that this topic occupies an important place in Grothendieck's SGA (Demazure–Grothendieck seminar).

The language of Chevalley's course "Fondement de la Géometrie Algébrique" (cf. [C1]) and seminar on Picard varieties (cf. [SC2]) is, of course, of the pre-Grothendieck era. In the "Fondement" the varieties are defined over algebraically closed fields and there are no rationality questions. There are no universal domains as in the Foundations of A. Weil and, in spirit, the language is close to that of Serre's paper FAC (Faisceaux algébriques cohérents), though there were no sheaves.

I can't refrain from narrating an amusing personal anecdote. During my stay in Paris I was living at the Cité Universitaire at what is called the U.S. house. During the last year of my stay I wanted to change to the German house. For this I was asked to produce a letter from my "patron" to the Director of the U.S. house. Presumably, it was a routine thing. I approached Chevalley. He wrote such a good letter for me that the Director of the U.S. house refused me permission to leave the house!

I returned to India by the end of the academic year 1959–60. On my way back, I visited Florence, bearing in mind a remark or advice of Chevalley that a visit to Florence is a must for understanding European culture. In

1967 when I was returning from a visit to the U.S. I stopped in Paris. Chevalley invited me and my wife to his house during this visit. Thereafter I had very few contacts with Chevalley.

All through my association I could sense that Chevalley had great human qualities. It has indeed been a great privilege to have worked with this great man at close quarters.

2. A well-known structure theorem of Chevalley which he announced in 1953 states that if G is a group variety, there is a uniquely determined normal connected affine (algebraic) subgroup such that G/H is an abelian variety. His proof was based on the theory of Picard varieties but for quite sometime he didn't publish his proof. The first published proofs of his theorem are due (independently) to Barsotti and Rosenlicht (see the references in [C3]). That Chevalley didn't publish his results soon may not come as a surprise, as he was known not to hurry with his publications. There could have been also another reason. The theory of Picard varieties was still evolving at that time. For his proof he needed a more general theory of Picard varieties than was available at that time. Chevalley also liked to understand and present everything in his own way. All these he did subsequently in his seminar on Picard varieties (cf. [SC2] and especially his paper [C2] which, as I mentioned above, was the construction presented at the seminar) and the paper [C3] giving the proof of his structure theorem (presumably on the lines of his original proof). In fact, even his course "Fondements de la Géometrie Algébrique" (loc. cit) seemed to be a part of his plan, as the language and basic results of algebraic geometry used in his seminar and the papers are based on his course and all these together constitute a self-contained entity.

Chevalley's proof of his structure theorem can be summarized as follows: Recall that for a variety X, the Albanese variety $\mathrm{Alb}(X)$ is an abelian variety which is universal for rational maps of X into abelian varieties. If, moreover X is smooth, $\mathrm{Alb}(X)$ is universal for morphisms of X into abelian varieties. Let G be a group variety (i.e., a connected, reduced algebraic group over an algebraically closed field). Then the Albanese variety $A = \mathrm{Alb}(G)$ of G is an abelian variety and one sees that the canonical map $G \longrightarrow A$ is a homomorphism of algebraic groups. Let H be the kernel of this homomorphism (with its reduced structure as a closed subgroup of G). Then it is not difficult to perceive that $\mathrm{Alb}(H)$ is trivial, i.e., H has zero *irregularity*. Then the crux of the proof is to show that if H is a group variety with zero irregularity, then H is an affine algebraic group. To see this, recall that the Picard variety of a variety X (if it exists) is the structure of a variety on the set of divisor classes on X, which are algebraically equivalent to zero. At that time it was known that the Picard variety of a smooth complete variety X is the *dual* (i.e., the Picard variety) of the Albanese variety, a discovery attributed to Chow. Then supposing that we have a theory of Picard varieties for a good class of *noncomplete* varieties (H may not be complete, in fact, a posteriori it is affine), it would follow that the Picard variety of H is trivial. Hence if D is an effective divisor on H and we denote by D_h the translate of D by an element h of H, D_h is linearly equivalent to D. Hence all D_h are in the complete linear system $|D|$. If H were complete, this would be a projective space of dimension r. Let us pretend that this is the case. Then through translations by elements of H, we could get a homomorphism $\rho_D : H \longrightarrow \mathrm{PGL}(r)$. Then by choosing a suitable finite number of D's, we would get an injective homomorphism of H into a finite product of $\mathrm{PGL}(r)$, which would imply that H is affine, etc. To make this intuitive proof work, one has therefore to work out a suitable theory of Picard varieties for noncomplete varieties. Chevalley does this for what he calls *semi-complete* normal varieties (a variety is *semi-complete* if the space of global sections for any fractionary sheaf of ideals is finite dimensional). He shows that the Picard variety of a semi-complete normal variety X is an abelian variety and if, moreover X is smooth, it is the dual of its Albanese variety (this is found in [C2] and was presented in the seminar [SC2]). The group variety is not semi-complete but roughly speaking can be embedded in one which is smooth and semi-complete and it turns out that this suffices.

3. I shall now give a brief idea of Chevalley's construction of Picard varieties and the contents of his seminar on Picard varieties.

Due to the work of Cartier (whose thesis appeared around 1957–58), it was realized that for general (i.e., not necessarily smooth) varieties, to have a good theory of divisors one has to work with Cartier divisors (i.e., locally principal divisors). Chevalley's construction of Picard varieties is based on Cartier divisors. In the literature, the seminar on Picard varieties [C5] and the paper [C2] are among the earliest to contain a systematic exposition of Cartier divisors, e.g., families of Cartier divisors, functorial properties, as well as some natural results on families of Cartier divisors (or divisor classes) on complete varieties which are nowadays proved by appealing to semi-continuity theorems on cohomology.

Let X be a variety with a family of Cartier divisor classes on X (or what we would call nowadays a family of line bundles) parametrized by a variety T (such that for a base point $t_0 \in T$, the corresponding line bundle on X is trivial). Then the canonical set theoretic map $T \longrightarrow \mathrm{Pic}(X)$ is said to be *algebraic*. If, moreover, T is a group variety and the map is a homomorphism of (abstract) groups, the map is said to be an *algebraic homomorphism.* According to Chevalley, the Picard variety of X is a universal object (if it exists) for all algebraic homomorphisms. The modern definition of Picard varieties is that of a universal object for all algebraic maps. Chevalley proves that the Picard variety of X exists if it is *semi-complete* and normal; further, it is an abelian variety and is a universal object for all algebraic maps. Granting the existence of the Picard variety of X, then that it is universal for all algebraic maps and also complete (hence an abelian variety), are deduced from the following result. Let $f : T \longrightarrow \mathrm{Pic}(X)$ be an algebraic map (with the condition about base points), T being a smooth curve but not necessarily complete. Then if J is the Jacobian of the complete curve containing T and $\chi : T \longrightarrow J$ the canonical map, f factors through an algebraic homomorphism $J \longrightarrow \text{Pic}(X)$. We see that this proof is close, in spirit, to the modern one based on the valuative criterion for properness. A systematic exposition of the construction of the Jacobian was given in the seminar and is found in [C2].

The proof of the existence of the Picard variety of X (normal and semi-complete) is motivated by the principle (mentioned above) that the Picard variety is the dual of the Albanese variety. Chevalley proves the existence of a "strict" Albanese variety for *any* variety X, i.e., there is an abelian variety which has the universal property for morphisms from X into abelian varieties, whereas, as we saw above, the classical $\text{Alb}(X)$ is a universal object for rational maps of X into abelian varieties. Then he shows the existence of the Picard variety of an abelian variety A. For this he uses the fact that there is a surjective map $C \longrightarrow A$, where C is a product of smooth complete curves. One sees the existence of the Picard variety of C as an easy consequence of that of the Picard variety (Jacobian) of a smooth complete curve. Then the existence of the Picard variety of A follows by a general argument. Finally, the Picard variety of the semi-complete normal variety X is shown to be the Picard variety of the strict Albanese variety of X (cf. [C2]).

As I mentioned above, Chevalley suggested to me the problem of constructing the Picard variety of a complete variety (not necessarily normal). By that time one felt that using Cartier divisors (or line bundles) the earlier theory for the case of smooth varieties would generalize to this case. In the last exposés of the seminar on Picard varieties (cf. [SC2]), Serre gave a construction of the strict Albanese varieties (and some generalisations) using Cartier's theory (Cartier operators etc.). Inspired by this, I found that by Cartier's descent theory (i.e., descent theory for line bundles for purely inseparable maps) one could prove the existence of the Picard variety of any complete variety X (in the sense of Chevalley, i.e., satisfying the universal property for algebraic homomorphisms). I required, in addition, the existence of the "field of rationality" for a Cartier divisor class on X. This was proved by Chevalley (cf. exposé 7, [5]). To prove that the Picard variety of X satisfies the universal property for all the algebraic maps required some effort. Let $f : T \longrightarrow \text{Pic}(X)$ be an algebraic map such that T is a smooth algebraic curve (and $f(t_0)$ is trivial for some $t_0 \in T$). One cannot expect that f would factorise through the Jacobian of the complete curve containing T, as the Picard variety of X is, in general, not complete. However, assuming that the Picard variety exists, one realises that f has to factorize through some "generalized Jacobian of T" (this is not uniquely determined). This meant that one had to prove that the map f has a "module" (see Serre's book [JPS]). Again, I profited by a remark of Serre. It was realized around that time that Rosenlicht's generalized Jacobian is, in fact, the Picard variety of a suitable complete nonnormal curve (cf.

Serre's book [JPS]). Serre told me that the construction of the Picard variety of a complete nonnormal curve generalizes to give the construction of the Picard variety of a complete nonnormal variety with only point singularities. Then I found that I could also check this. This suggested the possibility of proving the existence of the "module for the map f" in two steps, one for which the nonnormal singularities are point singularities and the other when the singularities are of dimension ≥ 1. Loosely speaking the proof was achieved along these lines. The case when the nonnormal singularity is of dimension ≥ 1 was handled by an induction process, using hyperplane sections (cf. [SC2]).

There is another approach to the construction of Picard varieties or schemes, which is the one usually given nowadays and is also classical. Let, for example, X be a projective scheme over a field. Then the Picard scheme of X is the quotient of an open subset of a suitable Hilbert scheme (classically a Chow scheme) associated to X, by a proper equivalence relation (in fact, it is a fibration by projective spaces). This could be called the Chow–Matsusaka–Grothendieck method. By this method Grothendieck first obtained existence theorems of Picard schemes involving nilpotent elements. A variant of this approach is to use Geometric Invariant Theory of D. Mumford. The methods of the Picard variety seminar, on the other hand, do not involve serious "projective methods" and give stronger results in the classical case. They had some influence in the development of powerful results on the representability of functors, e.g., the work of J. P. Murre (cf. [JPM]), where he shows the representability of the Picard functor of a proper algebraic scheme. The best approach now for the construction of Picard schemes by nonprojective methods is in the work of M. Artin (cf. [A]). One may consult ([BLR]) for recent developments on Picard schemes.

4. All through my stay in Paris, I was hardly aware of the great work of Chevalley on linear algebraic groups. In the early 60's I attempted a study of the Chevalley seminar on algebraic groups (cf. [SC1]) by giving a series of seminar talks, but it was only later when working on Standard Monomial Theory (SMT) that I understood and appreciated this aspect of his work. Chevalley's theory is a basic starting point for SMT. I benefitted by reading his influential and posthumously published work "Sur les décompositions cellularies des espaces G/B" (cf. [C5]). If my memory is correct, I obtained a copy of this paper from Iwahori when I visited Bonn in the early 70's. Many of the results on Schubert varieties in G/B, which are now standard, are to be found in this paper. For example, the dimension of a Schubert variety $X(w)$ in G/B, associated to an element w of the Weyl group of G is the *length* $\ell(w)$ of w (i.e., the length of a "reduced decomposition" of w in terms of simple reflections) and $X(w_1) \subseteq X(w_2)$ if and only if $w_1 \leq w_2$, i.e., a reduced decomposition of w_1 is a subword of a reduced decomposition of w_2. Often this is called the "Bruhat order" on the Weyl group, but as was pointed out at the American Math. Soc. Summer Institute on Algebraic Groups in 1991, this should be called the Chevalley order. In this paper Chevalley also proves his well-known formula on the multiplicity of intersection of a codimension one Schubert variety and on arbitrary Schubert variety. This formula is of importance in SMT. When A. Borel was writing up his talk on Chevalley's work at the Hyderabad Conference on Algebraic Groups (cf. [B]) and couldn't trace his copy of this paper of Chevalley, I was glad that I could send him one.

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References

[A] M. Artin, *Algebraization of formal moduli* I, In: Global Analysis, Papers presented in honour of K. Kodaira, University of Tokyo Press, Princeton University Press, 1970, pp. 21–71.

[B] A. Borel, *The work of Chevalley in Lie groups and algebraic groups*, In: Proceedings of the Hyderabad Conference on Algebraic Groups, published for the National Board for Higher Mathematics by Manoj Prakashan, Madras, 1991, pp. 1–22.

[BLR] S. Bosch, W. Lütkebohmert, M. Raynaud, *Néron Models*, Ergebn. der Math. un ihrer Grenzg., vol. 21, Springer, 1990.

[C1] C. Chevalley, *Fondements de la Géometrie Algébrique*, Secrétariat Mathématique, 11 Rue Pierre Curie, Paris 5, 1958.

[C2] C. Chevalley, *Sur la théorie de la variété de Picard*, Amer. J. Math. **82** (1960), 435–490.

[C3] C. Chevalley, *Une démonstration d'un théorème sur les groupes algébriques*, J. de Math. **39** (1960), 307–317.

[C4] C. Chevalley, *Certains schémas en groupes semi-simples*, Sém. Bourbaki, 1960/61, Exp. 219, 13e année 1960/61, 2e edition, May 1961.

[C5] C. Chevalley, *Sur les décompositions cellularies des espaces G/B* (With a foreword by A. Borel), In: Proc. Sympos. Pure Math., vol. 56, part 1, Algebraic Groups and Their Generalizations: Classical Methods (University Park, PA, 1991), Amer. Math. Soc., Providence, RI, 1994, pp. 1–23.

[JPM] J. P. Murre, *On contravariant functors from the category of preschemes over a field into the category of abelian groups* (*with an application to the Picard functor*), Publ. Math. I.H.E.S. **23** (1964), 581–619.

[SC1] Séminaire C. Chevalley, *Classifications des Groupes de Lie Algébriques*, vol. 1 and 2, Secrétariat Mathématique, 11 Rue Pierre Curie, Paris 5, 1958.

[SC2] Séminaire C. Chevalley, *Variétés de Picard*, Secrétariat Mathématique, 11 Rue Pierre Curie, Paris 5, 1960.

[JPS] J.-P. Serre, *Algebraic Groups and Class Fields* (*originally a course given in* 1956–57), Graduate Texts in Mathematics, vol. 117, Springer Verlag, 1988.