AFFINE GROUP SCHEMES AND HOPF ALGEBRAS

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ABSTRACT. The aim of these notes is to prove the equivalence between category of affine group schemes over algebraically closed field k and category of commutative, co-commutative Hopf algebras over k. We will explain in detail the notations and terminology, before giving proof of this theorem. Reader is assumed to know basic algebra.

CONTENTS

1. BASIC NOTIONS

1.1. Categories and Functors.

Definition 1. A category $\mathcal{C}$ is a class $Obj(\mathcal{C})$, called objects of the category and for every $X, Y \in Obj(\mathcal{C})$, a set $Hom_{\mathcal{C}}(X,Y)$, called set of morphisms from X to Y and a map (called composition) for every $A, B, C \in Obj(\mathcal{C})$:

$$Hom(A,B) \times Hom(B,C) \to Hom(A,C)$$

such that the following conditions are satisfied:

(C1) For $A, B, A', B' \in Obj(\mathcal{C})$, such that $(A,B) \neq (A',B')$, the sets $Hom(A,B)$ and $Hom(A',B')$ are disjoint.

(C2) There exists an element $1_A \in Hom(A,A)$ for every $A \in Obj(\mathcal{C})$, which acts as identity with respect to composition, i.e, for every $u \in Hom(A,B)$ (resp. $v \in Hom(B,A)$), $1_A u = u$ (resp. $v 1_A = v$), where $B \in Obj(\mathcal{C})$.

(C3) Composition is associative. That is for any $A, B, C, D \in Obj(\mathcal{C})$ and $u \in Hom(A,B)$, $v \in Hom(B,C)$ and $w \in Hom(C,D)$, we have

$$w(vu) = (wv)u$$

We also have a notion of opposite category to a category $\mathcal{C}$, which is denoted by $\mathcal{C}^{op}$. This category has same objects as $\mathcal{C}$ and morphisms

$$Hom_{\mathcal{C}^{op}}(A, B) = Hom_{\mathcal{C}}(B, A)$$

and

$$Hom_{\mathcal{C}^{op}}(A, B) \times Hom_{\mathcal{C}^{op}}(B, C) \to Hom_{\mathcal{C}^{op}}(A, C)$$

is obtained by composition

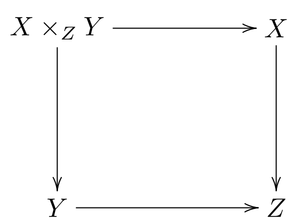
$$Hom_{\mathcal{C}}(C, B) \times Hom_{\mathcal{C}}(B, A) \to Hom_{\mathcal{C}}(C, A)$$

Definition 2. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. By a (covariant) *functor* F between $\mathcal{C}$ and $\mathcal{D}$, we understand a map $F : \mathcal{C} \to \mathcal{D}$, which to each object $A \in \mathcal{C}$, associates an object $F(A) \in \mathcal{D}$ and to each morphism $f \in Hom_{\mathcal{C}}(A, B)$, associates a morphism $F(f) \in Hom_{\mathcal{D}}(F(A), F(B))$, such that following axioms are satisfied:

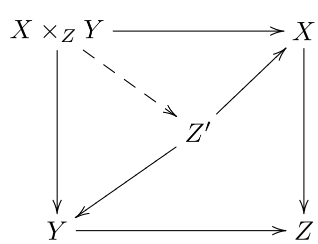
(F1) $F(1_A) = 1_{F(A)}$, for any $A \in \mathcal{C}$.
(F2) $F(uv) = F(u)F(v)$, where $v \in Hom_{\mathcal{C}}(A, B)$ and $u \in Hom_{\mathcal{C}}(B, C)$

Similarly, one can define contravariant functor by reversing arrows. More explicitly, for $f \in Hom_{\mathcal{C}}(A, B)$, we obtain $F(f) \in Hom_{\mathcal{D}}(F(B), F(A))$. Second axiom is thus changed accordingly. Note that a contravariant functor $F : \mathcal{C} \to \mathcal{D}$ is same as a covariant functor $F : \mathcal{C}^{op} \to \mathcal{D}$.

Definition 3. Let $X, Y \in Ob(\mathcal{C})$ and assume that $X \to Z$ and $Y \to Z$ are two morphisms in $\mathcal{C}$. The *fibre product* of X and Y over Z, denoted by $X \times_Z Y$, is defined to be an object in $\mathcal{C}$, together with morphisms $X \times_Z Y \to X$ and $X \times_Z Y \to Y$, such that following diagram is commutative:



Moreover, if there is another such object, say Z' and morphisms $Z' \to X$ and $Z' \to Y$, then there is a unique map $X \times_Z Y \to Z'$ such that the following diagram commutes:



1.2. **Presheaves and sheaves.**

1.3. **Local ringed spaces.**

Definition 4. A *ringed space* is a topological space X together with sheaf of rings $\mathcal{O}_X$, called *structure sheaf* of X. We say $(X, \mathcal{O}_X)$ is *local ringed space*, if for every $x \in X$, the ring of stalks over x, $\mathcal{O}_{X,x}$ is a local ring. We denote by $\mathfrak{m}_x$, the unique maximal ideal of $\mathcal{O}_{X,x}$.

In order to make sense of category of local ringed spaces, we need to define the notion of morphism between ringed spaces.

Definition 5. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two local ringed spaces. By a morphism between X and Y, we mean a continuous map (of topological spaces) $f : X \to Y$, together with morphism of sheaves $F : \mathcal{O}_Y \to f_*(\mathcal{O}_X)$.

2. Affine Schemes

Let A be a commutative ring with unity. We begin by associating a local ringed space to A, as follows:

(1) As topological space *Spec*(A) is collection of all prime ideals of A, with closed sets of the form

$$V(\mathfrak{a}) = \{\mathfrak{p} \mid \mathfrak{a} \subset \mathfrak{p} \text{ and } \mathfrak{p} \text{ is prime ideal of } A\}$$

where $\mathfrak{a}$ is any ideal of A. Note that for any ideal $\mathfrak{a}$ of A, the corresponding closed set $V(\mathfrak{a})$ is just $Spec(A/\mathfrak{a})$ and for any $f \in A$, the corresponding open set $Spec(A) \setminus V(f)$, denoted by $D(f)$, is just $Spec(A_f)$. Such open sets form basis for the above topology on *Spec*(A) and are usually called *distinguished* open sets or *principal* open sets.

(2) The structure sheaf $\mathcal{O}_{Spec(A)}$ is defined as (on distinguished open sets):

$$\mathcal{O}_{Spec(A)}(D_f) := A_f, \text{ for every } f \in A$$

For any $\mathfrak{p} \in Spec(A)$, the stalk $\mathcal{O}_{Spec(A),\mathfrak{p}}$ is just $A_\mathfrak{p}$, localization of A at prime $\mathfrak{p}$ and hence $(Spec(A), \mathcal{O}_{Spec(A)})$ is local ringed space.

Definition 6. Let A and B be two commutative rings with unity and let $f : A \to B$ be unital homomorphism of rings. We can define ${}^a f : Spec(B) \to Spec(A)$ by

$$f^*(\mathfrak{q}) = f^{-1}(\mathfrak{q})$$

which can be checked easily to be continuous map of topological spaces. Moreover, the morphism of sheaves $\tilde{f} : \mathcal{O}_{Spec(A)} \to^a f_* \mathcal{O}_{Spec(B)}$ is given by

$$\begin{array}{ccc}
\mathcal{O}_{Spec(A)}(D(h)) & \longrightarrow & \mathcal{O}_{Spec(B)}(D(h')) \\
\Vert & & \Vert \\
A_h & \longrightarrow & B_{h'}
\end{array}$$

where $h' = f(h)$ for any $h \in A$.

Thus we have defined *Spec* as a *contravariant functor* from category of commutative rings to category of local ringed spaces.

Definition 7. Let $(X, \mathcal{O}_X)$ be a local ringed space. We say X is *affine scheme* , if there exists a commutative ring with unity, A such that there is following isomorphism of local ringed spaces.

$$(X, \mathcal{O}_X) \cong (Spec(A), \mathcal{O}_{Spec(A)})$$

Note that a map between affine schemes is morphism if and only if it corresponds to ring homomorphism of corresponding rings. That is, *Spec* defined following equivalence of categories:

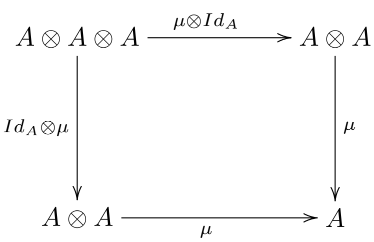
$$\{\text{commutative rings with unity}\} \quad \leftrightarrow \quad \{\text{affine schemes}\}$$

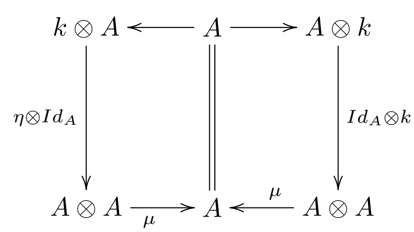
3. Affine schemes over k

Let k be algebraically closed field.

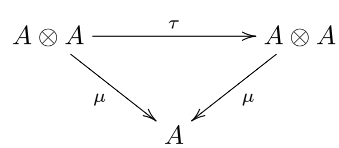
3.1. **Category of affine schemes over k.**

Definition 8. Let A be vector space over k. We say A is a *unital* k-algebra if there are k linear maps $\mu : A \otimes_k A \to A$ and $\eta : k \to A$, such that the following diagrams are commutative:





Moreover, we say A is commutative if the following diagram is commutative:



where τ is *flip map* ($\tau(a \otimes b) = b \otimes a$).

From now on, we will assume all k-algebras to be unital.

Definition 9. Let $(Spec(A), \mathcal{O}_{Spec(A)})$ be an affine scheme. We say it is affine scheme over k if A is finitely generated commutative k-algebra. We will often abuse notation and write $Spec(A)$ for corresponding affine scheme over k.

Note that by the way of construction we have the following equivalence of categories (or more precisely, anti-equivalence, since *Spec* is contravariant).

$$\{\text{Finitely generated commutative } k\text{-algebras}\}^{op} \quad \leftrightarrow \quad \{\text{Affine schemes over } k\}$$

We denote the category of affine schemes over k by $\mathbf{AffSch}_k$ and the category of finitely generated commutative k -algebras by $\mathbf{FGAlg}_k$.

3.2. **Fibre products in category AffSch$_k$.** In this section, our primary tool is the fact that *Spec* is a contravariant functor. We will explore the technique of *arrow reversal* that will be extremely useful in understanding language of coalgebras, bialgebras and Hopf algebras later.

Let A and B be finitely generated unital commutative k-algebras. We have the following natural maps (see Definition 8)

$$A \otimes A \to A$$

$$k \to A$$

$$\begin{array}{ccc} A & \longrightarrow & A \otimes B \\ \uparrow & & \uparrow \\ k & \longrightarrow & B \end{array}$$

By applying *Spec* to last diagram, we get the following commutative diagram of affine schemes:

$$\begin{array}{ccc} Spec(A) & \longleftarrow & Spec(A \otimes B) \\ \downarrow & & \downarrow \\ Spec(k) & \longleftarrow & B \end{array}$$

which allows us to identify (see Definition 3).

$$Spec(A \otimes B) = Spec(A) \times_{Spec(k)} Spec(B) \tag{1}$$

Moreover, we get the following natural morphisms of affine schemes:

$$Spec(A) \longrightarrow Spec(A) \times Spec(A) \tag{2}$$

$$Spec(A) \longrightarrow Spec(k) \tag{3}$$

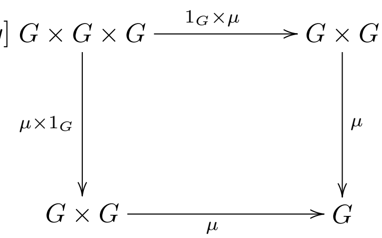
The morphism (2) is called *diagonal map* and (3) is called *structural morphism.* For most of the times, we will use these morphisms without explicit mention. Moreover, since for any k-algebra, A, $k \otimes A$ and $A \otimes k$ are canonically isomorphic to A (via scalar multiplication), we get the following canonical isomophism

$$Spec(A) \cong Spec(k) \times Spec(A) \cong Spec(A) \times Spec(k)$$

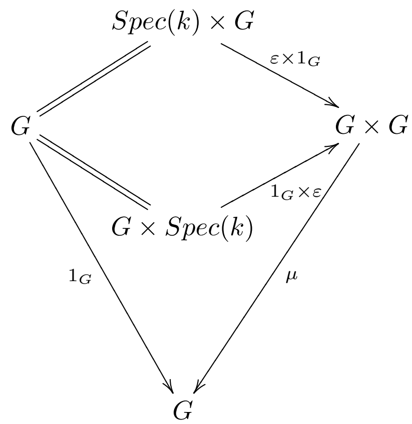
4. Affine group schemes

Definition 10. Let G be an affine scheme over k. We say G is *affine group scheme* over k, if there are morphisms $\mu : G \times G \to G$, $\gamma : G \to G$ and $\varepsilon : Spec(k) \to G$, such that the following diagrams are commutative:

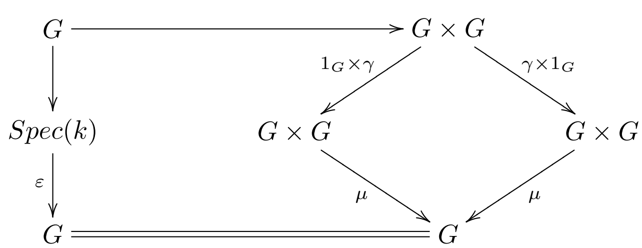
(4) [*Associativity*]



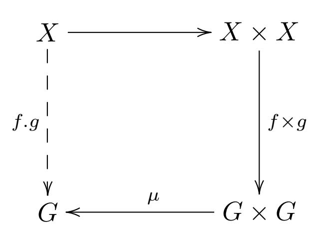
(5) [*Unit*]



(6) [*Inverse*]



Theorem 1. *Let G be affine group scheme over k. Then for every $X \in \boldsymbol{AffSch}_k$, $Hom(X, G)$ is a group under the multiplication (of $f, g \in Hom(X, G)$) defined by commutativity of following diagram:*



and corresponding to each $X \to Y$, morphism of affine schems over k, the induced map

$$Hom(Y, G) \to Hom(X, G)$$

is homomorphism of groups. In other words, we have a contravariant functor $Hom(-, G) : \boldsymbol{AffSch}_k \to Groups$.
Moreover, the above property characterizes affine group schemes over k. That is,

an affine scheme over k, say G is affine group scheme over k if and only if the functor $Hom(-, G)$ takes values in category of groups.

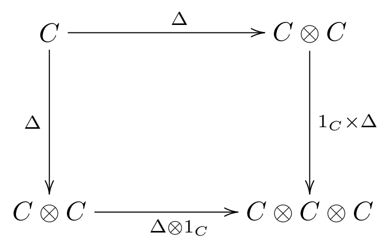
We denote by $\mathbf{AffGpSch}_k$, category of affine group schemes over k.

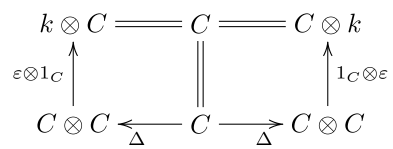
4.1. **General group objects.**

Definition 11. Let $\mathcal{C}$ be a category. Let $X \in \mathcal{C}$. We say X is a *group object* in $\mathcal{C}$ if $Hom(-, X)$ defines a contravariant functor from $\mathcal{C}$ to category of groups.

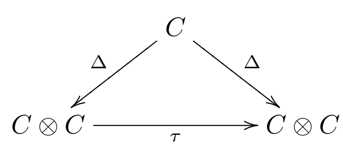
5. Hopf algebras

Definition 12. Let C be a k-vector space. We say that C is a *coalgebra* over k if there are k-linear maps $\Delta : C \to C \otimes C$ and $\varepsilon : C \to k$ such that the following diagrams are commutative:

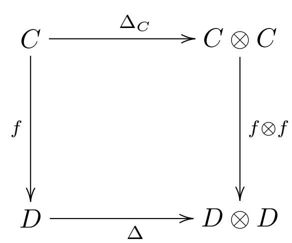


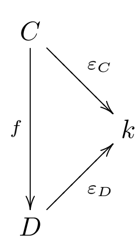


Moreover, we say that C is *co-commutative* if the following diagram commutes:

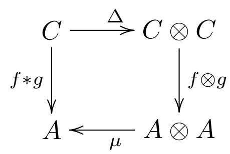


If C and D are two coalgebras over k, a k-linear map $f : C \to D$ is said to be coalgebra map if the following diagrams commute:





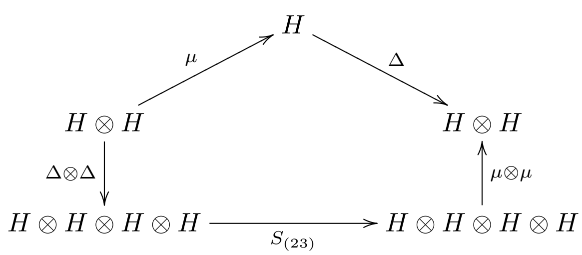
Now let A be a k-algebra and C be a k-coalgebra. Then $Hom_k(C, A)$ has canonical structure of k-vector space. We define a product on it, making it a unital k-algebra. For any two $f, g \in Hom(C, A)$, define *convolution product* $f * g$ by commutativity of following diagram:



It is easy to check that the composition $C \to k \to A$ (namely $\eta\varepsilon$) is unit with respect to convolution product.

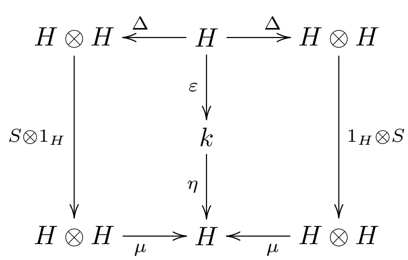
Definition 13. A *bialgbera* over k is tuple $(B, \mu, \eta, \Delta, \varepsilon)$ such that

(1) (B, μ, η) is k-algebra.
(2) (B, Δ, ε) is k-coalgebra.
(3) The following diagram commutes:



which means that Δ is algbera map (or μ is coalgebra map).

Definition 14. Let $(H, \mu, \eta, \Delta, \varepsilon)$ be a bialgebra. If there exists a k-linear map $S \in Hom(H, H)$, such that $S * 1_H = 1_H * S = \eta\varepsilon$, that is,



then H is called Hopf algebra and S is called antipodal map.

A morphism between Hopf algebras is both algebra and coalgebra morphism, which commutes with antipodal map. Let $\mathfrak{H}$ denote the category of commutative, co-commutative Hopf algebras over k (which are finitely generated over k).

6. Main Theorem

Theorem 2. *There is an anti-equivalence of categories between category of affine group schemes over k, **AffGpSch**$_k$ and category of commutative, co-commutative*

Hopf algebras over k, $\mathfrak{H}$. This equivalence is given by following functors, quasi-inverse to each other:

$$\begin{array}{rcl} \boldsymbol{AffGpSch}_k & \longrightarrow & \mathfrak{H} \\ (Spec(A), \mathcal{O}_{Spec(A)}) & \longmapsto & \mathcal{O}_{Spec(A)}(Spec(A)) = A \end{array}$$

$$\begin{array}{rcl} \mathfrak{H} & \longrightarrow & \boldsymbol{AffGpSch}_k \\ H & \longmapsto & (Spec(H), \mathcal{O}_{Spec(H)}) \end{array}$$

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