QUANTUM DOUBLE BRUHAT CELLS

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ABSTRACT. These are notes from my talk on Quantum double Bruhat cells, in Tapas '06, Northeastern University.

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1. QUANTIZATION

In this section, we introduce the problem of quantization.

1.1. **Quantization of Poisson algebras.** Historically, quantization is described as a process to move from classical mechanics to quantum mechanics.

$$\begin{tikzcd} Classical\ Mechanics \arrow[r, rightsquigarrow] & Quantum\ Mechanics \end{tikzcd}$$

$$\begin{tikzcd} (M, C^\infty(M)) \arrow[r, rightsquigarrow] & (\mathcal{H}, \mathcal{B}(\mathcal{H})) \end{tikzcd}$$

In above schematic diagram, M is a Poisson manifold (i.e, M is a manifold such that $C^\infty(M)$ is Poisson algebra; see definition 1) which is phase space in classical mechanics. The process of quantization replaces M by a Hilbert space $\mathcal{H}$ and $C^\infty(M)$ by algebra of operators on $\mathcal{H}$, denoted by $\mathcal{B}(\mathcal{H})$. However, for longest time, there was no "canonical" way or well-defined definition of quantization.

First definition of "quantization" appeared in *Deformation theory and quantization-I,II* by Moshe Flato, C. Fronsdal, A. Lichnerowics and D. Sternheimer (Annals of Physics, vol. 111; 1978), which we discuss here. Let us begin with some definitions:

Definition 1. Let A be unital, commutative algebra over k. By *Poisson structure* on A, we mean a k bilinear map $\{,\} : A \times A \to A$ satisfying following axioms:

(PB1) $\{a, a\} = 0$ for every $a \in A$.
(PB2) $\{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} = 0$, for every $a, b, c \in A$.
(PB3) $\{a, bc\} = \{a, b\}c + \{a, c\}b$, for every $a, b, c \in A$.

Note that first two axioms imply that A is a Lie algebra with respect to $\{,\}$ as Lie bracket. Third axiom states that for every $a \in A$, $\{a, .\} : A \to A$ is a *derivation* of A.

Assuming A is Poisson algebra over k, let $\widetilde{A}$ be $k[[h]]$ module (here h is a formal variable and $k[[h]]$ is ring of formal power series in h with coefficients from k). Assume that there exists a product $*$ in $\widetilde{A}$, which makes it a $k[[h]]$-algebra. Moreover, assume the following conditions:

(Q1) $\widetilde{A} \cong A[[h]]$ as $k[[h]]$-module.
(Q2) $\widetilde{A}/h\widetilde{A} \cong A$ as k-algebra.
(Q3) $\{a, b\} \equiv \frac{a*b-b*a}{h}$ modulo $h\widetilde{A}$, for every $a, b \in A$.

Under these hypothesis, $\widetilde{A}$ is called *quantization* of A and A is called *quasi classical limit* of $\widetilde{A}$.

Remark 1. *Note that, quantization need not exist for arbitrary Poisson algebra. In fact there are examples of Poisson algebras which do not admit any quantization. Also, there is no canonical way to quantize a given Poisson algebra. That is, a given Poisson algebra can admit more than one non equivalent quantizations. For quantization of $C^\infty(M)$, where M is Poisson manifold, see* Deformation quantization of Poisson manifolds *by Maxim Kontsevich (Letters in Math. Physics, 2003).*

Remark 2. *Deformation quantization of a Poisson algebra depends on Hochschild cohomology of the algebra. In a vague sense, second Hochschild cohomology group of A "parametrizes" the quantizations of A and third cohomology group consists of "obstructions" to quantizations. For more precise treatment, see* On deformations of rings and algberas *by M. Gerstenhaber (Annals of Mathematics, vol 79, no. 1;1964). However, computing Hochschild cohomology of algebras, which are not finite dimensional as k vector space, is not an easy problem.*

1.2. Quantization of Lie bialgebras.

The problem of quantizing a Poisson manifold is extremely non-trivial (it got settled just three years ago). However, quantum groups (which are main objects of study in these notes) arise from another subclass, namely of *Poisson Lie groups.* Let me define, what Poisson Lie groups are:

Definition 2. Let G be a Poisson manifold and a Lie group (i.e, multiplication and inverse are smooth maps). We say G is *Poisson Lie group* if these two structures are compatible in following sense:

$$\{f, g\}(xy) = \{f \circ l_x, g \circ l_x\}(y) + \{f \circ r_y, g \circ r_y\}(x)$$

where $f, g \in C^\infty(G)$, $x, y \in G$ and l_x, r_x denote left, right multiplication by x.

Definition 3. Let $\mathfrak{g}$ be finite dimensional k-vector space together with following two maps (k-linear)

$$[,] : \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{g}$$
$$\delta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$$

Assume the following axioms hold:

Lie1 $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$, for every $X, Y, Z \in \mathfrak{g}$.

Then we say $(\mathfrak{g}, [,])$ is Lie algebra over k. If in addition we have,

Lie2 $(\mathfrak{g}^*, \delta^*)$ is Lie algebra over k (here $\delta^* : \mathfrak{g}^* \wedge \mathfrak{g}^* \to \mathfrak{g}^*$ is dual to δ).
Lie3 For every $X, Y \in \mathfrak{g}$, we have

$$\delta([X,Y]) = X.\delta(Y) - Y.\delta(X)$$

where $\mathfrak{g}$ acts on $\mathfrak{g} \wedge \mathfrak{g}$ via "adjoint action", i.e,

$$X.(Y \wedge Z) = [X,Y] \wedge Z + Y \wedge [X,Z]$$

Then we say $(\mathfrak{g}, [,], \delta)$ is Lie bialgebra over k.

Notion of Lie bialgebra stems from classical correspondence (Lie's theorem)

$$\begin{tikzcd}
\mathit{Lie\ Groups} \arrow[r, leftrightsquigarrow] & \mathit{Lie\ algebras} \\
\mathit{Poisson - Lie\ groups} \arrow[u, hook] \arrow[r, leftrightsquigarrow] & \mathit{Lie\ bialgebras} \arrow[u, hook]
\end{tikzcd}$$

The correspondence between Poisson-Lie groups and Lie bialgebras was first proved by V.G. Drinfeld (see Quantum Groups in Proc. ICM Berkeley, 1986).

To introduce notion of quantization of Lie bialgebras, we need two more definitions. Recall that *universal enveloping algebra* $U\mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ is defined by

$$U\mathfrak{g} = T\mathfrak{g}/J$$

where $T\mathfrak{g} = \oplus_{n=0}^{\infty} \underbrace{\mathfrak{g} \otimes \ldots \otimes \mathfrak{g}}_{n-times}$, is tensor algebra of $\mathfrak{g}$ (free unital associative algebra over k vector space $\mathfrak{g}$) and J is two sided ideal generated by following terms:

$$X \otimes Y - Y \otimes X - [X,Y],\ \forall X, Y \in \mathfrak{g}$$

Definition 4. Let H be a k vector space, together with following k-linear maps

$$m : H \otimes H \to H \ \Delta : H \to H \otimes H$$

$$e : k \to H \ \varepsilon : H \to k$$

$$S : H \to H$$

satisfying following axioms:

(Hopf1) $(H, m, e(1))$ is unital, assiciative k-algebra.
(Hopf2) (H, Δ, ε) is co-unital, co-associative k-co-algebra. That is, following diagrams commute:

$$\begin{tikzcd}
H \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & H \otimes H \arrow[d, "1 \otimes \Delta"] \\
H \otimes H \arrow[r, "\Delta \otimes 1"'] & H \otimes H \otimes H
\end{tikzcd}$$

$$\begin{tikzcd}
& H \arrow[dl, equal] \arrow[dr, equal] & \\
k\otimes H & H\otimes H \arrow[l, "\varepsilon\otimes Id_H"'] \arrow[r, "Id_H\otimes\varepsilon"'] & H\otimes k
\end{tikzcd}$$

(Hopf3) Δ and ε are k-algebra maps.
(Hopf4) $m(S\otimes 1)(\Delta(X)) = m(1\otimes S)(\Delta(X)) = e(\varepsilon(X))$ for every $X\in H$.

Then we say $(H, m, e, \Delta, \varepsilon, S)$ is a Hopf algebra over k. Δ is called *co-multiplication*, ε is called *co-unit* and S is called antipode. Let $\sigma : H\otimes H \to H\otimes H$ be the flip map, i.e, $\sigma(a\otimes b) = b\otimes a$. We say H is *cocommutative* if

$$\Delta(a) = \sigma(\Delta(a)),\ \forall a\in H$$

Example 1. *Let $A = k[a, b, c, d]/(ad - bc - 1)$ with natural k algebra structure. Define the following maps:*

$$\Delta(a) = a\otimes a + b\otimes c;\ \Delta(b) = a\otimes b + b\otimes d$$
$$\Delta(c) = c\otimes a + d\otimes c;\ \Delta(d) = c\otimes b + d\otimes d$$
$$S(a) = d;\ S(b) = -b;\ S(c) = -c;\ S(d) = a$$
$$\varepsilon(a) = \varepsilon(d) = 1;\ \varepsilon(c) = \varepsilon(d) = 0$$

Then A becomes Hopf algebra (commutative) with Δ as comultiplication, ε as counit and S as antipode.

Example 2. *If $U\mathfrak{g}$ is universal enveloping algebra of Lie algebra $\mathfrak{g}$, then defining:*

$$\Delta(X) = X\otimes 1 + 1\otimes X,\ \forall X\in\mathfrak{g}$$
$$S(X) = -X,\ S(XY) = S(Y)S(X),\ X, Y\in\mathfrak{g}$$
$$\varepsilon(X) = 0,\ \forall X\in\mathfrak{g}$$

makes $U\mathfrak{g}$ into a Hopf algebra (cocommutative).

Now, take $(\mathfrak{g}, [,], \delta)$ to be Lie bialgebra over k. Assume that there exists a Hopf algebra H over $K = k[[h]]$, satisfying following axioms:

QUE1 $H\cong U\mathfrak{g}[[h]]$ as K-module.
QUE2 $H/hH\cong U\mathfrak{g}$ as Hopf algebras over k.
QUE3 $\delta(X)\equiv\frac{\Delta(X)-\sigma(\Delta(X))}{h}$ modulo (h).

Then we say H is quantization of Lie bialgebra $\mathfrak{g}$ or $\mathfrak{g}$ is quasiclassical limit of H.

Remark 3. *Note that in example 1, the algebra A is coordinate ring of $SL_2(k)$ and Δ, ε and S correspond to matrix multiplication, identity matrix and inverse matrix respectively. There is a theorem, which says that coordinate ring A of an affine variety X is Hopf algebra if and only if X is linear algebraic group.*

$$\begin{tikzcd}
\{f.g.\ commutative\ k-algebras\} \arrow[r, leftrightsquigarrow] & \{Affine\ k-varities\} \\
\{f.g.\ comm.\ Hopf\ k-algebras\} \arrow[u, hook] \arrow[r, leftrightsquigarrow] & \{Linear\ algebraic\ groups\} \arrow[u, hook]
\end{tikzcd}$$

(assuming k is algebraically closed).

Remark 4. *There is equivalent version for Hopf algebras of example 2.*

Every cocommutative Hopf algebra is
universal enveloping algebra
of some Lie algebra.

Remark 5. *The two versions of quantization introduced in this section are in fact dual to each other. More explicitly, if G is Poisson Lie group, $\mathfrak{g}$ is corresponding Lie bialgebra, then quantization of $C^\infty(G)$ can be achieved by taking* restricted dual *of quantization of $U\mathfrak{g}$.*

2. Some examples

In this section, we take $k = \mathbb{C}$, field of complex numbers.

2.1. **Standard Lie bialgebra structure.**

Let $\mathfrak{g} = \mathfrak{sl}_2$. It is well known (and easy to check) that $\mathfrak{g}$ is generated, as k vector space, by three elements:

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix};\ F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix};\ H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The Lie bracket on $\mathfrak{g}$ is given by:

$$[H,E] = 2E;\ [H,F] = -2F;\ [E,F] = H$$

(here for two matrices A and B, $[A,B] := AB - BA$). Define $\delta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$, as:

$$\delta(H) = 0;\ \delta(E) = \frac{1}{2} H \wedge E;\ \delta(F) = \frac{1}{2} H \wedge F$$

Reader should verify that δ defines a Lie bialgebra structure on $\mathfrak{g}$.

More generally, if $\mathfrak{g}$ is finite dimensional semisimple Lie algebra over k, given by Cartan Matrix $A = (a_{ij})_{1 \le i,j \le r}$ and $d_1, \dots, d_r$ are positive integers, satisfying

$$d_i a_{ij} = d_j a_{ji}$$

Then *standard Lie bialgebra* structure on $\mathfrak{g}$ is given by

$$\delta(H_i) = 0;\ \delta(E_i) = \frac{d_i}{2} H_i \wedge E_i;\ \delta(F_i) = \frac{d_i}{2} H_i \wedge E_i$$

for $1 \le i \le r$, where $(E,F_i,H_i)_{1 \le i \le r}$ are Chevalley generators. (see Appendix 1).

2.2. **Quantization of standard Lie bialgebras.**

Consider $\mathfrak{g} = \mathfrak{sl}_2$, together with standard Lie bialgebra structure introduced in previous section. We define $U_h(\mathfrak{sl}_2)$ as Hopf $k[[h]]$-algebra, as follows:

(Generators) E, F, H

(Relations) $HE - EH = 2E$, $HF - FH = -2F$

$$EF - FE = \frac{e^{hH} - e^{-hH}}{e^h - e^{-h}}$$

where $e^X = \sum_{n=0}^{\infty} \frac{X^n}{n!}$.

To make $U_h(\mathfrak{sl}_2)$ a Hopf algebra, we need to define comultiplication, counit and antipode.

$$\Delta(H) = H \otimes 1 + 1 \otimes H$$

$$\Delta(E) = e^{hH} \otimes E + E \otimes 1;\ \Delta(F) = 1 \otimes F + F \otimes e^{-hH}$$

$$S(E) = -e^{-hH} E;\ S(F) = -F e^{hH};\ S(H) = -H$$

$$\varepsilon(E) = \varepsilon(F) = \varepsilon(H) = 0$$

Reader should verify that substituting $h = 0$, gives us $U(\mathfrak{sl}_2)$ as Hopf algebra. Moreover,

$$\frac{\Delta(E) - \sigma(\Delta(E))}{h} = (H \otimes E - E \otimes H) + h(\cdots) \equiv H \wedge E,\ mod\ (h)$$

Thus we recover standard Lie bialgebra structure on $\mathfrak{sl}_2$ (upto a scalar).

Note that it is more convenient to make the substitution $e^h = q$ (change of variables). The algebra thus obtained is denoted by $U_q(\mathfrak{g})$. When $\mathfrak{g}$ is any finite dimensional simple Lie algebra over k, together with standard Lie bialgebra structure defined in previous section, we can give explicit description of $U_q(\mathfrak{g})$ (as Hopf $\mathbb{Q}(q)$-algebra) as follows (see Appendix for notations):

(Generators) $E_1, \ldots, E_r, F_1, \ldots, F_r$ and K_α, for every $\alpha \in Q$.

(Relations) $K_\alpha K_\beta = K_{\alpha+\beta}$, for $\alpha, \beta \in Q$.

$$K_\alpha E_i = q^{(\alpha,\alpha_i)} E_i K_\alpha;\ K_\alpha F_i = q^{-(\alpha,\alpha_i)} F_i K_\alpha$$

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_{\alpha_i} - K_{-\alpha_i}}{q^{d_i} - q^{-d_i}}$$

$$\sum_{l=0}^{l=1-a_{ij}} (-1)^l \left(\begin{array}{c} 1 - a_{ij} \\ l \end{array} \right)_{q^{d_i}} E_i^{1-a_{ij}-l} E_j E_i^l = 0$$

where $\left(\begin{array}{c} n \\ m \end{array} \right)_p = \frac{[n]_p!}{[m]_p! [n-m]_p!}$

$$[n]_p = \frac{p^n - p^{-n}}{p - p^{-1}};\ [n]_p! = [1]_1 \ldots [n]_p$$

This quantized algebra has following Hopf-structure:

$$\Delta(K_\alpha) = K_\alpha \otimes K_\alpha$$

$$\Delta(E_i) = K_{\alpha_i} \otimes E_i + E_i \otimes 1$$

$$\Delta(F_i) = 1 \otimes F_i + F_i \otimes K_{-\alpha_i}$$

$$S(E_i) = -K_{-\alpha_i} E_i;\ S(F_i) = -F_i K_{\alpha_i};\ S(K_\alpha) = K_{-\alpha}$$

$$\varepsilon(E_i) = \varepsilon(F_i) = 0;\ \varepsilon(K_\alpha) = 1$$

3. QUANTUM GROUPS AND QUANTUM DOUBLE BRUHAT CELLS

3.1. **Quantum Groups.** The following is taken from *Quantum Groups* by V.G. Drinfeld (in Proc. ICM, Berkeley, 1986). Recall that there is a general principle in mathematics.

$$\{Spaces\} \leftrightarrow \{Algebras\}$$

This becomes a theorem, when we take appropriate notion of space and algebra. For example, by Gelfand-Naimark theorem, category of locally compact topological spaces is dual to category of commutative C^*-algebras, via the correspondence: $X \longmapsto C_c(X)$, set of complex valued continuous functions on X with compact support. Also, category of affine varieties over k is dual to category of finitely generated commutative unital algebras over k. Now "group-spaces" correspond to "Hopf algebras".

In a similar fashion, one can define category of *quantum spaces* as dual to category of associative (not necessarily commutative) algebras and category of *quantum groups* as dual to category of (neither commutative nor cocommutative) Hopf algebras.

Remark 6. *It is important to note that quantum groups are not groups.*

The quantizations considered in previous section are all examples of quantum groups. In next section we describe Hopf dual of $U_q(\mathfrak{g})$, as quantized ring of regular functions.

3.2. **Quantized ring of functions.** Take U to be $U_q(\mathfrak{g})$, Hopf algebra over $\mathbb{Q}(q)$. Take $U^* = Hom_{\mathbb{Q}(q)}(U, \mathbb{Q}(q))$. U^* has canonical algebra structure induced from comultiplication in U. That is, given $f, g \in U^*$ and $u \in U$, one defines:

$$(fg)(u) = (f \otimes g)(\Delta(u))$$

Moreover, U^* also has structure of $U - U$ bimodule, as

$$(XfY)(u) = f(YuX)$$

where $X, Y, u \in U$ and $f \in U^*$. Now we define Hopf dual of U, as certain subalgebra of U^*.

$$U^\circ := \{f \in U^* | f(I) = 0 \text{ for some finite codimension Hopf ideal } I \text{ of } U\}$$

Given $\gamma, \delta \in P$ (see appendix for notations), one defines $P \times P$ weight space of U°, denoted by $\mathcal{O}_q(G)_{\gamma,\delta}$

$$\mathcal{O}_q(G)_{\gamma,\delta} := \{f \in U^\circ | K_\alpha f K_\beta = q^{(\alpha,\delta)+(\beta,\gamma)} f, ; \forall \alpha, \beta \in Q\}$$

Finally, take

$$\mathcal{O}_q(G) = \bigoplus_{\gamma,\delta \in P} \mathcal{O}_q(G)_{\gamma,\delta}$$

We briefly mention some properties of $\mathcal{O}_q(G)$:

1. $\mathcal{O}_q(G)$ is $P \times P$-graded subalgebra of U°.
2. $\mathcal{O}_q(G)$ is a domain.

3.3. **Quantum double Bruhat cells.** Quantum double Bruhat cells are obtained from $\mathcal{O}_q(G)$ in two steps, which imitate the classical result about describing double Bruhat cells via vanishing/non-vanishing of certain minors. Firstly, we define an ideal J, such that $\mathcal{O}_q(G)/J$ is quantized ring of functions on $\overline{G^{u,v}}$. Then we define certain elements of $\mathcal{O}_q(G)$ as *quantum minors*, which form a *multiplicatively closed subset* of $\mathcal{O}_q(G)/J$. The quantum double Bruhat cells are obtained as localizing at this subset.

Step 1: Given $u, v \in W$, take any reduced expression $u = s_{i_1} \dots s_{i_{l(u)}}$ and $v = s_{j_1} \dots s_{j_{l(v)}}$. Consider the sequence

$$\mathbf{i} = (-i_1, \dots, -i_{l(u)}, j_1, \dots, j_{l(v)})$$

For each i, take U_i to be subalgebra of U generated by $K_\alpha : \alpha \in Q$ and E_i and take U_{-i} to be subalgebra of U generated by $K'_\alpha s$ and F_i. Set

$$U_{\mathbf{i}} := U_{-i_1} \dots U_{-i_{l(u)}} U_{j_1} \dots U_{j_{l(v)}}$$

Let $J = \{f \in \mathcal{O}_q(G) | f(U_{\mathbf{i}}) = 0\}$.

Step 2: Assuming the notation introduced in previous step, let $\lambda \in P^+$. For each k, $1 \leq k \leq l(u)$, define

$$\eta_k := s_{i_{l(u)}} \dots s_{k+1}(H_k)$$

For each k, $1 \leq k \leq l(v)$, define

$$\zeta_k = s_{j_{l(v)}} \dots s_{j_{k+1}}(H_k)$$

Moreover, we define an element of $\mathcal{O}_q(G)$, denoted by Δ^λ, as follows:

$$\Delta^\lambda(FK_\alpha E) = \varepsilon(F) q^{(\lambda,\alpha)} \varepsilon(E)$$

where F (resp. E) is an element in subalgebra of U generated by $F_i's$ (resp. $E_i's$).

Step 3: Finally define *quantum minors* as

$$\Delta_{u\lambda,v\lambda} := \left(\frac{F_{j_1}^{\lambda(\zeta_1)}}{[\lambda(\zeta_1)]_{q^{d_{j_1}}}!} \cdots \frac{F_{j_{l(v)}}^{\lambda(\zeta_{l(v)})}}{[\lambda(\zeta_{l(v)})]_{q^{d_{j_{l(v)}}}}!} \right) .\Delta^\lambda \left(\frac{E_{i_{l(u)}}^{\lambda(\eta_{l(u)})}}{[\lambda(\eta_{l(u)})]_{q^{d_{i_{l(u)}}}}!} \cdots \frac{E_{i_1}^{\lambda(\eta_1)}}{[\lambda(\eta_1)]_{q^{d_{i_1}}}!} \right)$$

Step 4 Let $\pi : \mathcal{O}_q(G) \to \mathcal{O}_q(G)/J =: \mathcal{O}_q(\overline{G^{u,v}})$ be the canonical projection. Set

$$D_{u,v} := \{q^k \pi(\Delta_{u\lambda,\lambda}).\pi(\mu, v^{-1}(\mu)) | k \in \mathbb{Z}, \lambda, \mu \in P^+\}$$

Quantum double Bruhat cells $\mathcal{O}_q(G^{u,v})$ are defined as $\mathcal{O}_q(\overline{G^{u,v}})[D_{u,v}^{-1}]$.

Appendix

In this appendix, we review the basic theory of finite dimensional simple Lie algebras over $\mathbb{C}$. Let $A = (a_{ij})_{1 \leq i,j \leq r}$ be $r \times r$ matrix with integer entries, satisfying following conditions:

(CM1) $a_{ii} = 2$, for each i, $1 \leq i \leq r$.
(CM2) $a_{ij} \leq 0$, for $i \neq j$.
(CM3) There exists a diagonal matrix $D = (d_1, \dots, d_r)$, such that DA is symmetric, each d_i is positive integer and $(d_1, \dots, d_r)$ are coprime.
(CM4) A is non-degenerate.

Such a matrix A is called *Cartan matrix*. Given a Cartan matrix, one can define a Lie algebra $\mathfrak{g}(A)$, with generators E_i, F_i, H_i, $1 \leq i \leq r$ (called Chevalley generators) and relations:

$$[H_i, E_j] = a_{ij} E_j;\ [H_i, F_j] = -a_{ij} F_j$$

$$[E_i, F_j] = \delta_{ij} H_i;\ [H_i, H_j] = 0$$

$$(ad(E_i))^{1-a_{ij}} E_j = 0,\ i \neq j$$

$$(ad(F_i))^{1-a_{ij}} F_j = 0,\ i \neq j$$

In this setup, define $\mathfrak{h} = \oplus \mathbb{C} H_i$, abelian sub-Lie algebra generated by $H_1, \dots, H_r$. Since A is non-degenerate, there exist unique elements $\alpha_i \in \mathfrak{h}^*$, $1 \leq i \leq r$, given by

$$\alpha_i(H_j) = a_{ij}$$

These $\alpha_i's$ are called *simple roots*. Let $Q = \sum \mathbb{Z}\alpha_i$ be lattice generated by $\{\alpha_i,\ 1 \leq i \leq r\}$ inside $\mathfrak{h}^*$. Q is called *root lattice*. We define an inner product on $\mathfrak{h}^*$, by requiring

$$(\alpha_i, \alpha_j) = d_i a_{ij}$$

Define $P := \{\lambda \in \mathfrak{h}^* | \lambda(H_i) \in \mathbb{Z}\}$, called *weight lattice*. Note that $Q \subset P$, but not necessarily equal. Also, with respect to the inner product defined above, we have:

$$\lambda \in P,\ \alpha \in Q \Rightarrow (\lambda, \alpha) \in \mathbb{Z}$$

Now for each i, define an element s_i of $Aut(\mathfrak{h}^*)$ as follows:

$$s_i(\lambda) = \lambda - \lambda(H_i)\alpha_i$$

We define $W \subset Aut(\mathfrak{h}^*)$ as subgroup generated by simple reflections $s_1, \ldots, s_r$. W is called *Weyl group* of $\mathfrak{g}$. Given $w \in W$, let $w = s_{i_1} \ldots s_{i_l}$ be a smallest expression for w in terms of $s_i's$. We define $l(w) = l$ for any such smallest expression (called length of w).

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