Worksheet Number Thirteen

Continued Fractions

We are all familiar with decimal numbers, and know that a repeating decimal gives a rational number, while one that never repeats and goes on forever represents an irrational number. There is another way to represent real numbers called continued fractions. This idea was used by the Indian mathematician Aryabhata in his book, the Aryabhatia, to solve linear indeterminate Diophantine equations in 499 A.D. He worked near the current city of Patna in Bihar, in northern India.

Here is the idea, with the Euclidean algorithm next to the construction of the continued fraction, to show the relationship:

$$\frac{11}{7} = 1 + \frac{4}{7} \qquad 11 = 1(7) + 4$$

$$= 1 + \frac{1}{\frac{7}{4}} = 1 + \frac{1}{1 + \frac{3}{4}} \qquad 7 = 1(4) + 3$$

$$= 1 + \frac{1}{1 + \frac{1}{\frac{4}{3}}} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{3}}} \qquad 4 = 1(3) + 1 \text{ and then } 3 = 3(1) + 0.$$

Now you calculate in the same way the continued fraction for $\frac{13}{5}$.

Calculate the continued fraction for $\frac{170}{53}$.

Calculate the continued fraction for $\frac{53}{41}$

Now, here is how Aryabhata used these continued fraction to solve for one solution of the Diophantine equation 19X + 51Y = 1.

Represent $\frac{51}{19}$ as a continued fraction :

$$\frac{51}{19} = 2 + \frac{13}{19} = 2 + \frac{1}{\frac{19}{13}} = 2 + \frac{1}{1 + \frac{6}{13}} = 2 + \frac{1}{1 + \frac{1}{\frac{13}{6}}} = 2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{6}}}.$$

Now go backwards and calculate the fraction we just got, but leave out the last fraction down to the right, the $\frac{1}{6}$;

$2 + \frac{1}{1 + \frac{1}{2}} = 2 + \frac{2}{3} = \frac{8}{3}$. We cross multiply and subtract the original

$\frac{51}{19}\,and\,\frac{8}{3}$ thus: 51(3) – 19(8) = 153 – 152 = 1. So X = 3 and Y = -8 gives a solution.

Use continued fractions to find one solution of the Diophantine equation 53X + 41Y = 1.

For math credit, use continued fractions to find a solution to the Diophantine equations

a. 364X + 227Y = 1

b. 172X + 20Y = 1,000.