Worksheet Number Fifteen

Amicable Numbers and Thabit ibn Qurra

We have seen the Pythagorean concepts of **perfect** and **amicable** numbers earlier in this course. The last theorem in book IX of Euclid, **Theorem IX.36** states that if $2^{n+1}-1$ is a prime number then $2^n \cdot (2^{n+1}-1)$ is a perfect number. Examples are 6, 28, 496 and 8,128. This is the pinnacle of ancient Greek number theory.

The only amicable numbers known to the ancient Greeks were 220 and 284. To find some formula similar to the one in Euclid, but for amicable numbers, must occur to some students of number theory. A successful such mathematician was **Thabit ibn Qurra**, who grew up in Harran, in what is now Turkey, and lived from about 836 to 901. He ended up being the court astronomer in Baghdad.

Thabit's theorem giving a formula to possibly construct infinitely many pairs of amicable numbers is the following:

Theorem. Suppose that n > 1 is an integer and that $p_n = 3 \cdot 2^n - 1$ and $q_n = 9 \cdot 2^{2n-1}$-1 have the property that $p_n,\ q_n$ *and* p_{n-1} are prime numbers. Then a = $2^n \cdot p_n \cdot p_{n-1}$ and $b = 2^n \cdot q_n$ are amicable numbers, with a an abundant number and b a deficient number.

An **abundant** number has the sum of its proper divisors bigger than the number, like 12. A **deficient** number has thee sum of its divisors smaller than the number, like 9.

$p_1 =$

Calculate $p_2 =$

and $q_2 =$

What do you get for a and b? Are they amicable? Abundant or deficient?

$p_3 =$

Calculate $p_4 =$ and I inform you that 1151 is a prime number.

and $q_4 =$

What do you get for a and b? How about abundance and deficiency?
Check that they are amicable:

$$\text{Calculate} \quad \begin{aligned} p_5 &= \\ p_6 &= \\ p_7 &= \\ \text{and } q_7 &= \end{aligned} \quad \text{and I inform you that 73,727 is a prime number.}$$

What new a and b can you get here? Is there a problem with p_5?

On a different scale of accomplishment, a sixteen-year-old Italian student, Nicolo I. Paganini, in 1866, discovered the second smallest pair of amicable numbers. They are 1184 and 1210.

a, Check that they are amicable.

b. Check that they cannot come from Thabit's formula.

c. Is one abundant and the other deficient?

This Nicolo Paganini is not the composer and virtuoso violinist Nicolo Paganini (1782-1840) who wrote his first sonata at the age of 8.