Worksheet Number Fourteen

A Quadratic Diophantine Equation
(Pell's Equation, a misnomer)

We have seen **Aryabhata's** continued fraction production of solutions to the two-variable, linear, indeterminate Diophantine equation ax + by = c. We have seen **Brahmagupta's** classification of all solutions to such equations, which he wrote about in his book the **Brahmasputa Siddhanta**, written in 628 A.D. in Bhimal, Rajasthan, India.

Here is a next problem: find integer solutions to the equation $y^2 = ax^2 + 1$.

Brahmagupta, in the Brahmasputa Siddhanta, makes a big start on solving this problem. He bypasses finding a first solution in an algorithmic way, and proceeds by trial and error to find one solution, other than (x, y) = (0, 1) or (x, y) = (0, -1). **This is his only gap.**

Suppose we are looking at $y^2 = 92x^2 + 1$.

Brahmagupta lets x = 1 and then needs to add 8 instead of 1 to get a perfect square, so he does this. (x, y) = (1, 10) satisfies the equation $y^2 = 92x^2 + 8$.

He does this because he has an incredible method by which he can do a weird multiplication of two known solutions to get a third solution to equations of this type. If (x, y) = (1, 10) is a solution for 8 in place of 1, then you can get a new x equal to 2xy and a new y equal to $92x^2 + y^2$. This new pair will be a solution for 64, in place of 8. That is, it will solve the equation $y^2 = 92x^2 + 64$.

If we try it we see that 2xy = 20 = the new x and the new y = 192. (192)(192) = 92(20)(20) + 64.

If you now divide this equation by 64, you get $\frac{y^2}{8^2} = 92\frac{x^2}{8^2} + 1$, which gives you 192/8 = 24 = y and x = 20/8 = 5/2 from the preceding calculation. You only need to do the "weird" multiplication on this solution to get an answer:

2xy = 120 = x and 92(25/4) + (24)(24) = 1151.

This will not always work for him.

But once he has a solution to $y^2 = ax^2 + 1$, **he can multiply to get infinitely many solutions**: Say $y^2 = 120x^2 + 1$. If x = 1 then y = 11 will work. (1, 11) multiplies by itself gives (22, 241). It works.

To multiply (x, y) times (u, v) you get (xv + uy ,120xu + yv). Thus (1, 11) times (22, 241) gives (483, 5291). Hmmmm. Try it on a calculator. It works. You can go on like this forever.

Brahmagupta is multiplying numbers like

$\left(3+5\sqrt{120}\right)\cdot\left(4+6\sqrt{120}\right)=(12+30\cdot 120)+(20+18)\cdot\sqrt{120}$ which amounts to algebra using numbers like a + b $\sqrt{120}$. This game really had more players in the 19th century, 1200 years after Brahmagupta lived.

Now begin to find infinitely many solutions to the equation $y^2 = 80x^2 + 1$. Find three, say.

Math credit: Find three solutions to the equation $y^2 = 83x^2 + 1$