The Arithmetic of the Hyperreal Numbers

<u>Moderate Number</u> will be used to mean the usual real numbers we are socialized to accept.

<u>Infinitesimal Number</u> will be used to denote numbers whose absolute value is greater than zero but smaller than any real number, or the negative of such a number.

<u>Infinite Number</u> will be used to denote numbers whose absolute value is greater than any real number, or the negative of such a number. These numbers are not Georg Cantor's transfinite cardinal numbers.

Suppose that m and n are positive, moderate numbers.

Suppose that t and s are non-zero infinitesimal numbers.

Suppose that B and H are positive infinite numbers.

1. Then notice that the difference of m and n, m – n can be moderate, like 12 – 7 = 5 or 7 – 12 = -5,

   or m – n can be infinitesimal like 12 – (12 + t) = -t.

2. Also check out that t/s can be moderate, infinitesimal or infinite:

   t/t = 1; $\frac{t}{t^2}$ = 1/t which is infinite; $\frac{t^2}{t}$ = t.

3. So can tH because it may be that t = 1/H so that tH = 1, or $t = \frac{1}{H^2}$,

   so tH = 1/H which is infinitesimal or H = $\frac{1}{t^2}$ so tH = 1/t which is infinite.

4. So can H – B, as it might happen that B = H – 1, so H – B = H – (H – 1) = 1, or B = H/2, so H – B = H/2, or B = H + t, so H – B = H = (H + t) = t

Practice problems

Give your reasons as to whether the following hyperreal numbers are moderate, infinite or infinitesimal:

1. $1 + t$
2. $1 + 100t$
3. $1 + 10{,}000t$
4. $1 – t$
5. $1 – 100t$
6. $B + 1$
7. $B – 100$
8. $B + 2 + t$
9. $B + 10 + s$
10. $B + 1/B$
11. $1 + \ t^7 \ \square \ 100t^{11}$
12. $\dfrac{7H^5}{H^5}$
13. $\dfrac{7H^5}{H^6}$
14. $\dfrac{7H^6}{H^5}$
15. $\dfrac{B^2 \ \square \ 1}{B + 1}$
16. Give three examples showing that H/B can be infinite, moderate or infinitesimal.