A Pythagorean geometric proof that $\sqrt{2}$ is irrational

Irrational refers to the fact that $\sqrt{2}$ cannot be expressed as a ratio of two integers. That is, there is no common measure of any integer and $\sqrt{2}$. If $\sqrt{2} = \frac{m}{n}$ in lowest terms, then m times $\frac{1}{n}$ gives $\sqrt{2}$ and $\frac{1}{n}$ would be such a common measure.

Anthyphairesis was an ancient Greek geometric form of the Euclidean algorithm. To look at the geometric and our numeric forms side by side, here is an example:

Find the G.C.D. of 49 and 35:

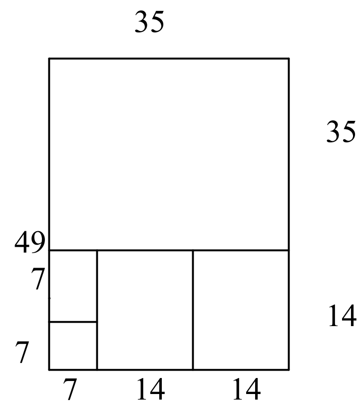
$49 = 1(35) + 14$

$35 = 2(14) + 7$

$14 = 2(7)$

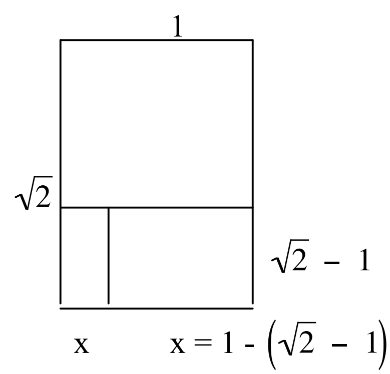
so 7 is the G.C.D.

A corresponding geometric Construction:



In the figure, the fact that there are the two 7X7 squares ultimately, shows that the 7 is a common measure of both (via 14) 35 and 49, and is the largest such common measure.

Now try to do this with 1 and $\sqrt{2}$. First we notice that $\left(\sqrt{2} - 1\right)\left(\sqrt{2} + 1\right) = 1$, so that $\sqrt{2} - 1 = \frac{1}{\sqrt{2} + 1}$. We now start the rectangles in anthyphairesis for 1 and $\sqrt{2}$.



In the large rectangle, the ratio of the sides is $\frac{\sqrt{2}}{1}$. Two stages down, the smallest rectangle has sides in the ratio

$$\frac{2 - \sqrt{2}}{\sqrt{2} - 1} = \frac{\left(2 - \sqrt{2}\right)\left(\sqrt{2} + 1\right)}{1} = \frac{\sqrt{2}}{1},$$

which is a similar rectangle. Now one sees that this similar rectangle is going to repeat every two steps, forever. There will never be a square, giving a common measure of 1 and $\sqrt{2}$.